

# Nonlinear Evolution of External Ideal MHD Modes Near the Boundary of Marginal Stability

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The nonlinear evolution of external ideal MHD-modes is determined from the equations of ideal MHD by employing a reductive perturbation method which uses a driving parameter for expansion. The reduction of the plasma equations is the same as for internal modes and was treated previously [1]. A main problem arising in addition for external modes is the reduction of the nonlinear boundary conditions. The set of reduced boundary conditions is obtained on the undisplaced boundary in the marginally stable equilibrium position. Another additional problem arises from the fact that the linear MHD operator is only selfadjoint for linear eigenmodes but not for the higher order mode corrections. This complicates the determination of nonlinear amplitude equations for the marginal mode which are obtained from solubility conditions. The amplitude equations are qualitatively the same as for internal modes. Quantitatively, the calculation of the coefficients in these is different. Explicit expressions for the coefficients are derived in full generality. The effect of higher order corrections to the nonlinear amplitude equations is discussed quantitatively for one of two possible cases and qualitatively for the other.

## 1. Introduction

In a previous paper [1] the nonlinear evolution of internal ideal MHD modes has been considered. As was mentioned there (note added in proof), a special symmetry case denoted in this paper “typical tokamak case” was also treated by Hu [2].

This paper deals with the corresponding theory for external modes. It is no straight forward extension of the internal mode problem since due to the nonlinearity of the boundary conditions additional problems arise. Specifically, the linear MHD operator  $\mathbf{F}$  which plays an important role in the set of reduced plasma equations, is selfadjoint for all perturbations occurring in the internal mode problem. However, in contrast to the opinion expressed in [2], this is no longer true for the nonlinear external mode problem: According to [3],  $\mathbf{F}$  is only selfadjoint in the subspace of linear eigenmodes which satisfy the linearized boundary conditions of ideal MHD while it is generally not selfadjoint with respect to higher order eigenmode corrections satisfying different boundary conditions.

A special case of nonlinear external mode evolution has been treated in [4] and [5]. There, under a series of simplifying assumptions, the evolution of external kink modes in a sharp boundary  $z$ -pinch with circular cross-section has been considered. The

approach of [4–5] is different from ours and appears not very apted for generalization. A list of further references concerning nonlinear plasma dynamics may be found in [1].

It is an essential assumption of this paper that the linear modes the nonlinear evolution of which is considered are nondegenerate regular modes with finite norm, i.e. that they belong to the discrete spectrum of  $\mathbf{F}$ . Since the reductive perturbation method employed yields always an amplitude equation for the nonlinear mode evolution, the main task is not the derivation of this amplitude equation but the determination of its coefficients.

The paper is organized as follows: In Section 2 we formulate the full nonlinear external mode problem, i.e. we collect all equations and boundary conditions which must be taken into account. In Sect. 3, we introduce the scaling of all quantities and reduce the dynamic plasma and vacuum equations by the reductive perturbation method. Since the plasma motion can be treated as for internal modes, all corresponding results are quoted from [1] to which we refer also with respect to the concept of driving parameters. Section 4 deals with the reduction of the boundary conditions. In Sect. 5, nonlinear amplitude equations for the marginal mode are derived and discussed. They have the same structure as those obtained for internal modes. General expressions for calculating the coefficients in these equations are presented in this section. Section 6

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discusses the effect of higher order corrections in connection with the problem of energy conservation. Concerning the plasma motion, the description used in this paper is Eulerian while concerning the boundary conditions we have adopted a Lagrangian description. In Sect. 7 finally an approach of the problem with a full Lagrangian description is sketched.

## 2. The Nonlinear External Mode Problem

We consider the nonlinear evolution of linear normal modes belonging to the discrete spectrum in free boundary plasmas surrounded by a vacuum region. The vacuum region can either extend towards infinity or be surrounded by a stabilizing wall. In this paper, we shall always assume the presence of a stabilizing wall with infinite conductivity. The case of an infinite vacuum region can then be treated by letting this wall move towards infinity. Current carrying conductors which may be present in the vacuum region for plasma containment are assumed to be completely permeable to magnetic field perturbations.

It is assumed that the equilibrium quantities  $p_0$ ,  $\mathbf{B}_0$  etc. depend on a driving parameter  $\lambda$  such that the plasma goes from stable to unstable when  $\lambda$  exceeds a critical value  $\lambda_0$ . The concept of driving parameters was discussed at length in [1].

In the case of external modes, the additional possibility exists that the position of the external wall depends on  $\lambda$ . Since many instabilities can be stabilized by an external conducting wall, taking as driving parameter the distance of the surface  $\mathfrak{C}_0$  of this is in many cases an especially convenient choice. Also, the shape and position of the plasma surface  $S$  may depend on  $\lambda$ . To make things not too complicated, we shall not consider this possibility in the present paper. We assume that all equilibrium quantities satisfy the relevant equilibrium equations and care only about the nonequilibrium dynamics or about the bifurcation of new equilibrium states.

Let us first consider boundary conditions. From the inhomogeneous ones of Maxwells equations, there come conditions which determine surface charges and surface currents. Important in MHD-theory are the currents

$$\mathbf{J} = \mathbf{n} \times [\mathbf{B}] \quad (1)$$

on the plasma surface  $S$  (normal  $\mathbf{n}$ ), where

$$[\mathbf{B}] = \mathbf{B}^\vee - \mathbf{B} \quad (2)$$

is the difference between the magnetic field on the vacuum- and plasma side of  $S$ .

According to Faradays law, in a coordinate system moving with the plasma surface  $S$  the tangential component of the electric field must be continuous across  $S$ . Since in the plasma  $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ , this condition is

$$\mathbf{n} \times \mathbf{E}^\vee = 0. \quad (3)$$

The electric field  $\mathbf{E}^\vee$  in the vacuum region is induced by the nonequilibrium magnetic field

$$\nabla \times \mathfrak{A} = \mathbf{B}^\vee - \mathbf{B}_0^\vee. \quad (4)$$

Setting

$$\mathbf{E}^\vee = -\partial \mathfrak{A} / \partial t \quad (5)$$

which due to  $\nabla \cdot \mathbf{E}^\vee = 0$  implies

$$\nabla \cdot \mathfrak{A} = 0, \quad (6)$$

we can rewrite (3) as

$$\mathbf{n} \times \partial \mathfrak{A} / \partial t = -\mathbf{n} \cdot \mathbf{v} \mathbf{B}^\vee \text{ on } S \quad (7)$$

like in the linear theory. Finally, from  $\nabla \cdot \mathbf{B} = 0$  the boundary condition

$$\mathbf{n} \cdot [\mathbf{B}] = 0 \quad (8)$$

is obtained. In equilibrium, we have  $\mathbf{n}_0 \cdot \mathbf{B}_0 = 0$ , and since the magnetic field is frozen into the plasma, we have  $\mathbf{n} \cdot \mathbf{B} = 0$  during the whole plasma motion. Locally, the electric boundary condition (3) can always be brought into the form

$$\nabla \psi \times \mathbf{E}^\vee = 0. \quad (9)$$

Since with (4–5) the divergence of this equation is

$$\nabla \psi \cdot \partial \mathbf{B}^\vee / \partial t = 0, \quad (10)$$

also the vacuum field remains tangential on  $S$  so that condition (8) is automatically satisfied and can be disregarded. Similarly as in the linear theory, there may be cases where (7) can be replaced by the simpler condition (8). However, it must be remembered that (8) contains less information than (7) and may lead to incorrect results [6]. We shall therefore use (7) exclusively in this paper.

From the equation of continuity, in a system moving with  $S$  the condition

$$\mathbf{n} \cdot \mathbf{v}' = 0 \quad (11)$$

is obtained. It may be considered as defining the plasma surface. We shall use the parameter repre-

sensation

$$\mathbf{r} = \mathbf{r}_{00}(u, v) \quad (12)$$

for the equilibrium position of the plasma surface  $S = S_{00}$  where the notation indicates that by assumption  $S_{00}$  does not depend on  $\lambda$ . The plasma displacement  $\xi(\mathbf{r}, t)$  is defined by

$$\partial \xi(\mathbf{r}, t) / \partial t = \mathbf{v}(\mathbf{r} + \xi(\mathbf{r}, t), t), \quad (13)$$

where  $\mathbf{v}(\mathbf{r}, t)$  is the plasma velocity. If we represent the moving plasma surface by

$$\mathbf{r} = \mathbf{r}_{00}(u, v) + \xi(\mathbf{r}_{00}(u, v), t), \quad (14)$$

condition (11) is satisfied automatically.

Frequently, for a moving surface rather the scalar representation

$$f(\mathbf{r}, t) = 0$$

is used which may be given dissolved with respect to one of three appropriately chosen coordinates. It could have been used in this paper as well. A condition corresponding to (13) follows from the requirement that a fluid element which was ever attached to the surface can only move along this. Since it causes some difficulties to find the connection between (13) and the corresponding condition in terms of  $f$ , a special example is considered in Appendix II.

From the momentum equation finally the condition

$$p + B^2/2 = B^v/2 \quad \text{on } S \quad (15)$$

is obtained.

On the surface  $\mathbb{C}_0$  of the conducting wall, corresponding to (3) we must have  $\mathbf{n} \times \mathbf{E}^v = 0$ . With (5) and after timeintegration this is

$$\mathbf{n} \times \mathfrak{A} = 0 \quad \text{on } \mathbb{C}_0. \quad (16)$$

In addition, (8) must be satisfied. Since again  $\mathbf{n} \cdot \mathbf{B}^v = 0$  follows from (16), and since  $\mathbf{B} = 0$  in the conductor, (8) can again be disregarded. In the plasma, the density  $\varrho$ , pressure  $p$ , current density  $\mathbf{j}$  etc. must satisfy the nonlinear equations

$$\begin{aligned} \varrho(\partial \mathbf{v} / \partial t + \mathbf{v} \cdot \nabla \mathbf{v}) &= \mathbf{j} \times \mathbf{B} - \nabla p, \quad \mathbf{j} = \nabla \times \mathbf{B}, \\ \partial \mathbf{B} / \partial t &= \nabla \times (\mathbf{v} \times \mathbf{B}), \\ \partial p / \partial t &= -(\mathbf{v} \cdot \nabla p + (5/3)p \nabla \cdot \mathbf{v}), \\ \partial \varrho / \partial t &= -\nabla \cdot (\varrho \mathbf{v}). \end{aligned} \quad (17)$$

In the vacuum region,  $\mathfrak{A}$  must satisfy the equations

$$\nabla \times (\nabla \times \mathfrak{A}) = 0, \quad \nabla \cdot \mathfrak{A} = 0. \quad (18)$$

The set of (17)–(18) must be supplemented by the boundary conditions (7) and (15) on  $S$  and (16) on  $\mathbb{C}_0$ .

### 3. Scaling of Plasma and Vacuum Quantities and Reduction of the Nonequilibrium Equations

#### 3.1. Plasma Region

Setting

$$\tau = \lambda - \lambda_0,$$

the equilibrium values  $\varrho_0$  of  $\varrho$ ,  $p_0$  of  $p$  and  $\mathbf{B}_0$  of  $\mathbf{B}$  are expanded

$$\begin{aligned} \varrho_0 &= \varrho_{00}, \\ p_0 &= p_{00} + \tau p_{01} + \tau^2 p_{02} + \tau^3 p_{03} + \dots, \\ \mathbf{B}_0 &= \mathbf{B}_{00} + \tau \mathbf{B}_{01} + \tau^2 \mathbf{B}_{02} + \tau^3 \mathbf{B}_{03} + \dots, \end{aligned} \quad (19)$$

where  $\nabla \varrho_0 \equiv 0$  and  $\partial \varrho_0 / \partial \lambda \equiv 0$  is assumed. The driving parameter  $\tau$  is itself expanded with respect to a small dimensionless parameter  $\varepsilon$ :

$$\tau = \varepsilon \tau_1 + \varepsilon^2 \tau_2 + \varepsilon^3 \tau_3 + \dots \quad (20)$$

It turns out that for external modes the same scaling of the nonequilibrium quantities is possible as the one for internal modes, i.e. we have the expansions

$$\begin{aligned} \varrho &= \varrho_{00} + \varepsilon \varrho_1 + \dots, \\ p &= p_{00} + \varepsilon(\tau_1 p_{01} + p_1) + \varepsilon^2(\tau_2 p_{01} + \tau_1^2 p_{02} + p_2) \\ &\quad + \varepsilon^3(\tau_3 p_{01} + 2\tau_1 \tau_2 p_{02} + \tau_1^3 p_{03} + p_3) + \dots, \\ \mathbf{B} &= \mathbf{B}_{00} + \varepsilon(\tau_1 \mathbf{B}_{01} + \mathbf{B}_1) + \varepsilon^2(\tau_2 \mathbf{B}_{01} + \tau_1^2 \mathbf{B}_{02} + \mathbf{B}_2) \\ &\quad + \varepsilon^3(\tau_3 \mathbf{B}_{01} + 2\tau_1 \tau_2 \mathbf{B}_{02} + \tau_1^3 \mathbf{B}_{03} + \mathbf{B}_3) + \dots. \end{aligned} \quad (21)$$

As for internal modes, we introduce a slow time  $T$  through

$$\partial / \partial T = \sqrt{|\tau|} \partial / \partial t \quad (22)$$

while again the velocity field  $\mathbf{v}$  is represented by

$$\mathbf{v} = \sqrt{|\tau|} \partial \boldsymbol{\Phi} / \partial T. \quad (23)$$

For  $\boldsymbol{\Phi}$  and the plasma shift  $\xi$  defined in (13) we make the expansions

$$\boldsymbol{\Phi} = \varepsilon \boldsymbol{\Phi}_1 + \varepsilon^2 \boldsymbol{\Phi}_2 + \varepsilon^3 \boldsymbol{\Phi}_3 + \dots, \quad (24)$$

$$\xi = \varepsilon \xi_1 + \varepsilon^2 \xi_2 + \varepsilon^3 \xi_3 + \dots \quad (25)$$

If (19)–(24) is inserted in (17), the latter are reduced in the same way as for internal modes. Thus, by analogy we may set

$$\boldsymbol{\Phi}_1 = A_1(T) \boldsymbol{\Phi}_1(\mathbf{r}), \quad (26)$$

and  $\Phi_1$  must again satisfy the marginal mode equation

$$\mathbf{F}_{00}(\Phi_1) = 0, \quad (27)$$

$\mathbf{F}_{00}$  being defined in (A8).  $\Phi_2$  must satisfy (40) of [1],

$$\mathbf{F}_{00}(\Phi_2) = \tau_1 \mathcal{Q}_{00} \partial^2 \Phi_1 / \partial T^2 - \tau_1 \mathbf{F}_{01}(\Phi_1) - (1/2) \mathbf{G}_{00}(\Phi_1, \Phi_1). \quad (28)$$

Since the right hand side contains terms  $\sim \tau_1 A_1$  and  $A_1^2/2$ , we can again make the ansatz

$$\Phi_2 = A_2(T) \Phi_{20}(\mathbf{r}) + \tau_1 A_1(T) \Phi_{21}(\mathbf{r}) + (1/2) A_1^2(T) \Phi_{22}(\mathbf{r}) \quad (29)$$

with

$$\mathbf{F}_{00}(\Phi_{20}) = 0. \quad (30)$$

It turns out that with this ansatz also the boundary conditions of the external mode problem can be satisfied

$\Phi_3$  must satisfy (57) of [1],

$$\begin{aligned} \mathbf{F}_{00}(\Phi_3) = & \mathcal{Q}_{00} \{ \tau_2 \ddot{A}_1 \Phi_1 + \tau_1 [\ddot{A}_2 \Phi_{20} + \tau_1 \ddot{A}_1 \Phi_{21} \\ & + \dot{A}_1^2 (\Phi_{22} + \Phi_1 \cdot \nabla \Phi_1) \\ & + A_1 \ddot{A}_1 (\Phi_{22} - \Phi_1 \cdot \nabla \Phi_1)] \} \\ & - \{ \tau_1 \mathbf{F}_{01}(\Phi_{20}) A_2 + \mathbf{G}_{00}(\Phi_1, \Phi_{20}) a_{12} \\ & + \mathbf{G}_{00}(\Phi_{20}, \Phi_1) a_{21} + [\tau_1^2 (\mathbf{F}_{01}(\Phi_{21}) \\ & + \mathbf{F}_{02}(\Phi_1)) + \tau_2 \mathbf{F}_{01}(\Phi_1)] A_1 \\ & + \tau_1 [\mathbf{F}_{01}(\Phi_{22}) + \mathbf{G}_{01}(\Phi_1, \Phi_1) \\ & + \mathbf{G}_{00}(\Phi_1, \Phi_{21}) + \mathbf{G}_{00}(\Phi_{21}, \Phi_1)] A_1^2/2 \\ & + [\mathbf{G}_{00}(\Phi_1, \Phi_{22}) + 2 \mathbf{G}_{00}(\Phi_{22}, \Phi_1) \\ & + \mathbf{H}_{00}(\Phi_1, \Phi_1, \Phi_1)] A_1^3/6 \}, \end{aligned} \quad (31)$$

where  $a_{12}$  and  $a_{21}$  are defined through

$$\dot{a}_{12} = \dot{A}_1 A_2, \quad \dot{a}_{21} = \dot{A}_2 A_1, \quad (32a)$$

$$a_{12} + a_{21} = A_1 A_2, \quad (32b)$$

(32b) being a consequence of (32a).

For the perturbational quantities  $p_1, \mathbf{B}_1, p_2, \mathbf{B}_2$ , the following results were obtained (Eqs. (38), (50), and (51) of [1]):

$$p_1 = A_1(T) \tilde{p}_1(\mathbf{r}), \quad \mathbf{B}_1 = A_1(T) \tilde{\mathbf{B}}_1(\mathbf{r}) \quad (33)$$

with

$$\tilde{p}_1 = \mathbf{p}_{10}^*(\Phi_1), \quad \tilde{\mathbf{B}}_1 = \mathbf{B}_{10}(\Phi_1) \quad (34)$$

\* Due to a change in the technics of printing, the letter used in [1] was no longer available. It should be born in mind that according to the definition (A4)  $\mathbf{p}$  is a scalar operator.

and

$$\begin{aligned} p_2 = & A_2(T) \tilde{p}_{20}(\mathbf{r}) + \tau_1 A_1(T) \tilde{p}_{21}(\mathbf{r}) \\ & + (1/2) A_1^2(T) \tilde{p}_{22}(\mathbf{r}), \\ \mathbf{B}_2 = & A_2(T) \tilde{\mathbf{B}}_{20}(\mathbf{r}) + \tau_1 A_1(T) \tilde{\mathbf{B}}_{21}(\mathbf{r}) \\ & + (1/2) A_1^2(T) \tilde{\mathbf{B}}_{22}(\mathbf{r}) \end{aligned} \quad (35)$$

with

$$\begin{aligned} \tilde{p}_{20} = & \mathbf{p}_{10}(\Phi_{20}),^* \\ \tilde{p}_{21} = & \mathbf{p}_{10}(\Phi_{21}) + \mathbf{p}_{11}(\Phi_1), \\ \tilde{p}_{22} = & \mathbf{p}_{10}(\Phi_{22}) + \mathbf{p}_1(\Phi_1, \mathbf{p}_{10}(\Phi_1)), \\ \tilde{\mathbf{B}}_{20} = & \mathbf{B}_{10}(\Phi_{20}), \\ \tilde{\mathbf{B}}_{21} = & \mathbf{B}_{10}(\Phi_{21}) + \mathbf{B}_{11}(\Phi_1), \\ \tilde{\mathbf{B}}_{22} = & \mathbf{B}_{10}(\Phi_{22}) + \mathbf{B}_1(\Phi_1, \mathbf{B}_{10}(\Phi_1)). \end{aligned} \quad (36)$$

Anticipating the second order result (91a), with (29) and (32b) the results (53)–(54) of [1] can be written

$$\begin{aligned} p_3 = & \mathbf{p}_{10}(\Phi_3) + (\tau_1 A_2 + \tau_2 A_1) \mathbf{p}_{11}(\Phi_1) \\ & + A_1 A_2 \mathbf{p}_1(\Phi_1, \mathbf{p}_{10}(\Phi_1)) \\ & + \tau_1^2 A_1 [\mathbf{p}_{11}(\Phi_{21}) + \mathbf{p}_{12}(\Phi_1)] \\ & + (1/2) \tau_1 A_1^2 [\mathbf{p}_{11}(\Phi_{22}) + \mathbf{p}_1(\Phi_1, \mathbf{p}_{10}(\Phi_{21})) \\ & + \mathbf{p}_1(\Phi_{21}, \mathbf{p}_{10}(\Phi_1)) + \mathbf{p}_1(\Phi_1, \mathbf{p}_{11}(\Phi_1))] \\ & + (A_1^3/6) [\mathbf{p}_1(\Phi_1, \mathbf{p}_{10}(\Phi_{22})) \\ & + 2 \mathbf{p}_1(\Phi_{22}, \mathbf{p}_{10}(\Phi_1)) \\ & + \mathbf{p}_1(\Phi_1, \mathbf{p}_1(\Phi_1, \mathbf{p}_{10}(\Phi_1)))] \end{aligned} \quad (37)$$

$$\begin{aligned} \mathbf{B}_3 = & \mathbf{B}_{10}(\Phi_3) + (\tau_1 A_2 + \tau_2 A_1) \mathbf{B}_{11}(\Phi_1) \\ & + A_1 A_2 \mathbf{B}_1(\Phi_1, \mathbf{B}_{10}(\Phi_1)) \\ & + \tau_1^2 A_1 [\mathbf{B}_{11}(\Phi_{21}) + \mathbf{B}_{12}(\Phi_1)] \\ & + (1/2) \tau_1 A_1^2 [\mathbf{B}_{11}(\Phi_{22}) + \mathbf{B}_1(\Phi_1, \mathbf{B}_{10}(\Phi_{21})) \\ & + \mathbf{B}_1(\Phi_{21}, \mathbf{B}_{10}(\Phi_1)) + \mathbf{B}_1(\Phi_1, \mathbf{B}_{11}(\Phi_1))] \\ & + (A_1^3/6) [\mathbf{B}_1(\Phi_1, \mathbf{B}_{10}(\Phi_{22})) \\ & + 2 \mathbf{B}_1(\Phi_{22}, \mathbf{B}_{10}(\Phi_1)) \\ & + \mathbf{B}_1(\Phi_1, \mathbf{B}_1(\Phi_1, \mathbf{B}_{10}(\Phi_1)))] \end{aligned}$$

Expanding the nonlinear equation (13), the relations

$$\begin{aligned} \xi_1 = & \Phi_1, \quad \xi_2 = \Phi_2 + (1/2) \Phi_1 \cdot \nabla \Phi_1, \\ \xi_3 = & \Phi_3 + A_1 A_2 \Phi_1 \cdot \nabla \Phi_1 \\ & + (1/2) \tau_1 A_1^2 (\Phi_{21} \cdot \nabla \Phi_1 + \Phi_1 \cdot \nabla \Phi_{21}) \\ & + (A_1^3/6) \left[ \Phi_{22} \cdot \nabla \Phi_1 + 2 \Phi_1 \cdot \nabla \Phi_{22} \right. \\ & \left. + (\Phi_1 \cdot \nabla \Phi_1) \cdot \nabla \Phi_1 + \Phi_1 \Phi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \Phi_1 \right] \end{aligned} \quad (38)$$

are obtained\*\*.

\*\* The relations (25) of [1] corresponding to (38) of this paper are wrong. They were corrected in [7]. The short proof of the correct relations is repeated here for the readers convenience.



Proof

Taylor expansion of (13) with respect to  $\xi$  yields

$$\begin{aligned} \partial \xi / \partial t &= \mathbf{v}(\mathbf{r}, t) + \xi \cdot \nabla \mathbf{v}(\mathbf{r}, t) \\ &+ (1/2) \xi \xi : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \mathbf{v}(\mathbf{r}, t) + \dots \end{aligned} \quad (39)$$

With (22–23) and the expansions (24–25), equating equal powers of  $\varepsilon$  on both sides of (39) we get

$$\begin{aligned} \partial \xi_1 / \partial T &= \partial \Phi_1 / \partial T, \\ \partial \xi_2 / \partial T &= \partial \Phi_2 / \partial T + \xi_1 \cdot \nabla \partial \Phi_1 / \partial T \\ \partial \xi_3 / \partial T &= \partial \Phi_3 / \partial T + \xi_2 \cdot \nabla \partial \Phi_1 / \partial T \\ &+ \xi_1 \cdot \nabla \partial \Phi_2 / \partial T + (1/2) \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \partial \Phi_1 / \partial T. \end{aligned} \quad (40)$$

From these equations,  $\xi_1$  in (38) follows immediately, and with this and (26),  $\xi_2$  follows as well. With the results (38) for  $\xi_1$ ,  $\xi_2$  and with (29) and (32) one obtains

$$\begin{aligned} \xi_2 \cdot \nabla \partial \Phi_1 / \partial T &= \frac{\partial}{\partial T} \{ a_{12} \Phi_{20} \cdot \nabla \Phi_1 \\ &+ (1/2) \tau_1 A_1^2 \Phi_{21} \cdot \nabla \Phi_1 \\ &+ (A_1^3/6) [\Phi_{22} \cdot \nabla \Phi_1 + (\Phi_1 \cdot \nabla \Phi_1) \cdot \nabla \Phi_1] \}, \\ \xi_1 \cdot \nabla \partial \Phi_2 / \partial T &= \frac{\partial}{\partial T} \{ a_{21} \Phi_1 \cdot \nabla \Phi_{20} \\ &+ (1/2) \tau_1 A_1^2 \Phi_1 \cdot \nabla \Phi_{21} + (A_1^3/3) \Phi_1 \cdot \nabla \Phi_{22} \}, \\ (1/2) \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \Phi_1 &= \frac{\partial}{\partial T} \left( (A_1^3/6) \Phi_1 \Phi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \Phi_1 \right). \end{aligned}$$

With this,  $\partial \xi_3 / \partial T$  in (40) can be integrated to yield  $\xi_3$  in (38), if in addition the second order result (91a) and (32b) are used.  $\square$

From (26), (29) and (38) we get

$$\xi_1 = A_1(T) \tilde{\xi}_1(\mathbf{r}), \quad \tilde{\xi}_1 = \Phi_1 \quad (41)$$

and

$$\begin{aligned} \xi_2 &= A_2(T) \tilde{\xi}_{20}(\mathbf{r}) + \tau_1 A_1(T) \tilde{\xi}_{21}(\mathbf{r}) \\ &+ (A_1^2(T)/2) \tilde{\xi}_{22}(\mathbf{r}) \end{aligned} \quad (42)$$

with

$$\begin{aligned} \tilde{\xi}_{20} &= \Phi_{20}, \quad \tilde{\xi}_{21} = \Phi_{21}, \\ \tilde{\xi}_{22} &= \Phi_{22} + (1/2) \Phi_1 \cdot \nabla \Phi_1. \end{aligned} \quad (43)$$

### 3.2. Vacuum region

The equilibrium field  $\mathbf{B}_0^v$  may depend on the driving parameter and, in agreement with (19), has

the expansion

$$\mathbf{B}_0^v = \mathbf{B}_{00}^v + \tau \mathbf{B}_{01}^v + \tau^2 \mathbf{B}_{02}^v + \tau^3 \mathbf{B}_{03}^v + \dots \quad (44)$$

The perturbational field (4) is scaled like  $\mathbf{B} - \mathbf{B}_0$  so that in agreement with (21)

$$\begin{aligned} \mathbf{B}^v &= \mathbf{B}_{00}^v + \varepsilon (\tau_1 \mathbf{B}_{01}^v + \mathbf{B}_1^v) \\ &+ \varepsilon^2 (\tau_2 \mathbf{B}_{01}^v + \tau_1^2 \mathbf{B}_{02}^v + \mathbf{B}_2^v) \\ &+ \varepsilon^3 (\tau_3 \mathbf{B}_{01}^v + 2 \tau_1 \tau_2 \mathbf{B}_{02}^v + \tau_1^3 \mathbf{B}_{03}^v + \mathbf{B}_3^v) + \dots \end{aligned} \quad (45)$$

and

$$\mathfrak{A} = \varepsilon \mathfrak{A}_1 + \varepsilon^2 \mathfrak{A}_2 + \varepsilon^3 \mathfrak{A}_3 + \dots \quad (46)$$

With (46), (18) are reduced to

$$\begin{aligned} \nabla \times (\nabla \times \mathfrak{A}_i) &= 0, \quad \nabla \cdot \mathfrak{A}_i = 0, \\ i &= 1, 2, 3, \dots \end{aligned} \quad (47)$$

Obviously, with the ansatz

$$\mathbf{B}_1^v = A_1(T) \tilde{\mathbf{B}}_1^v(\mathbf{r}), \quad \mathfrak{A}_1 = A_1(T) \tilde{\mathfrak{A}}_1(\mathbf{r}), \quad (48)$$

and

$$\begin{aligned} \mathbf{B}_2^v &= A_2(T) \tilde{\mathbf{B}}_{20}^v(\mathbf{r}) + \tau_1 A_1(T) \tilde{\mathbf{B}}_{21}^v(\mathbf{r}) \\ &+ (A_1^2(T)/2) \tilde{\mathbf{B}}_{22}^v(\mathbf{r}), \\ \mathfrak{A}_2 &= A_2(T) \tilde{\mathfrak{A}}_{20}(\mathbf{r}) + \tau_1 A_1(T) \tilde{\mathfrak{A}}_{21}(\mathbf{r}) \\ &+ (A_1^2(T)/2) \tilde{\mathfrak{A}}_{22}(\mathbf{r}), \end{aligned} \quad (49)$$

corresponding to (33) and (35), (47) can be satisfied.  $\mathfrak{A}_1$  and  $\tilde{\mathfrak{A}}_{2i}$  ( $i = 0, 1, 2$ ) must then be also solutions of these. It will be seen that the ansatz (48)–(49) agrees also with all boundary conditions.

### 4. Reduction of the Boundary Conditions

The boundary conditions (7) and (15) are posed on the displaced plasma surface (14). If the expansions and scalings (21)–(25) are inserted, in both cases functions

$$f(\mathbf{r}, t) = f_{00} + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \dots \quad (50)$$

must be evaluated at  $\mathbf{r} = \mathbf{r}_{00} + \xi$  for small  $\xi$ . This is most easily accomplished by first expanding  $f(\mathbf{r}_{00} + \xi, t)$  into a Taylor series with respect to  $\xi$  and then

inserting the expansions (25) and (50). The result is

$$\begin{aligned} f(\mathbf{r}_{00} + \xi, t) = & f_{00} + \varepsilon (f_1 + \xi_1 \cdot \nabla f_{00}) \\ & + \varepsilon^2 \left( f_2 + \xi_2 \cdot \nabla f_{00} + \xi_1 \cdot \nabla f_1 + (1/2) \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} f_{00} \right) \\ & + \varepsilon^3 \left( f_3 + \xi_3 \cdot \nabla f_{00} + \xi_2 \cdot \nabla f_1 + \xi_1 \cdot \nabla f_2 \right. \\ & + \xi_1 \xi_2 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} f_{00} + (1/2) \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} f_1 \\ & \left. + (1/6) \xi_1 \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} f_{00} \right), \end{aligned}$$

where all quantities on the right hand side are evaluated on the undisplaced plasma surface  $\mathbf{r}_{00}(u, v)$  and where

$$\begin{aligned} (1/2) \xi_1 \xi_2 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} f_{00} \\ + (1/2) \xi_2 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} f_{00} = \xi_1 \xi_2 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} f_{00} \end{aligned}$$

has been used.

#### 4.1. Condition $p + B^2/2 = B^{\vee 2}/2$ on $S$

First of all, the equilibrium quantities  $p_0$  and  $\mathbf{B}_0$  must satisfy

$$p_0 + B_0^2/2 - B_0^{\vee 2}/2 = 0 \quad (52)$$

on the undisplaced plasma surface (12). Inserting (19) and (44) in (52), with the definitions (A14)–(A17) we get the reduced equilibrium boundary conditions

$$[P_0] = 0, \quad i = 0, 1, 2, 3, \dots \quad (53)$$

If equilibria without surface currents  $\mathbf{J}_0$  are considered, we have

$$\mathbf{B}_0 = \mathbf{B}_0^{\vee} \quad (54)$$

on  $S_{00}$ . Setting

$$\mathbf{B}_0 = B_0 \mathbf{b}_0, \quad \mathbf{B}_0^{\vee} = B_0^{\vee} \mathbf{b}_0^{\vee} \quad (55)$$

with  $\mathbf{b}_0 = \mathbf{B}_0/B_0$  etc., (54) implies

$$B_0 = B_0^{\vee}, \quad \mathbf{b}_0 = \mathbf{b}_0^{\vee} \quad (56)$$

on the whole surface  $S_{00}$ . With this, from the equilibrium equations

$$\begin{aligned} \nabla(p_0 + B_0^2/2) = & \mathbf{B}_0 \cdot \nabla \mathbf{B}_0 = B_0^2 \mathbf{b}_0 \cdot \nabla \mathbf{b}_0 \\ & + \mathbf{b}_0 (\mathbf{b}_0 \cdot \nabla) B_0^2/2, \\ \nabla B_0^{\vee 2}/2 = & \mathbf{B}_0^{\vee} \cdot \nabla \mathbf{B}_0^{\vee} = B_0^{\vee 2} \mathbf{b}_0^{\vee} \cdot \nabla \mathbf{b}_0^{\vee} \\ & + \mathbf{b}_0^{\vee} (\mathbf{b}_0^{\vee} \cdot \nabla) B_0^{\vee 2}/2, \end{aligned} \quad (57)$$

and (56) we obtain

$$\nabla(p_0 + B_0^2/2 - B_0^{\vee 2}/2) = 0 \quad (58)$$

since  $\mathbf{b}_0 \cdot \nabla B_0^2/2 = \mathbf{b}_0^{\vee} \cdot \nabla B_0^{\vee 2}/2$  are derivatives in the plasma surface  $S_{00}$  and since the field line curvature  $\mathbf{b}_0 \cdot \nabla \mathbf{b}_0$  is continuous across this. Inserting (19) and (44) in (58), with (A14)–(17) we get the conditions

$$\nabla[P_0] = 0, \quad i = 0, 1, 2, 3, \dots \quad (59)$$

which are satisfied in addition to (53) for  $\mathbf{J}_0 = 0$ .

Turning now to the nonequilibrium case of condition (15), we set

$$f = p + B^2/2 - B^{\vee 2}/2 \quad (60)$$

and obtain the expansion functions  $f_{00}, f_1, f_2, \dots$  in (50) by inserting (21) and (45). In order to satisfy (15), the coefficients of all  $\varepsilon$ -powers in (51) must be put zero yielding the conditions

$$p_i + \mathbf{B}_{00} \cdot \mathbf{B}_i = \mathbf{B}_{00}^{\vee} \cdot \mathbf{B}_i^{\vee} + \xi_i \cdot \nabla [P_{00}] - \mathfrak{P}_i^*, \quad (61)$$

where

$$\mathfrak{P}_1 = 0,$$

$$\mathfrak{P}_2 = \tau_1 (\mathbf{B}_{01} \cdot \mathbf{B}_1 - \mathbf{B}_{01}^{\vee} \cdot \mathbf{B}_1^{\vee} - \xi_1 \cdot \nabla [P_{01}]) + B_1^2/2 - B_1^{\vee 2}/2$$

$$- \xi_1 \cdot \nabla [P_1] - (1/2) \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} [P_{00}],$$

$$\mathfrak{P}_3 = \tau_2 (\mathbf{B}_{01} \cdot \mathbf{B}_1 - \mathbf{B}_{01}^{\vee} \cdot \mathbf{B}_1^{\vee} - \xi_1 \cdot \nabla [P_{01}])$$

$$+ \tau_1 \left[ \mathbf{B}_{01} \cdot \mathbf{B}_2 - \mathbf{B}_{01}^{\vee} \cdot \mathbf{B}_2^{\vee} \right. \quad (62)$$

$$+ \tau_1 (\mathbf{B}_{02} \cdot \mathbf{B}_1 - \mathbf{B}_{02}^{\vee} \cdot \mathbf{B}_1^{\vee} - \xi_1 \cdot \nabla [P_{02}]$$

$$- \xi_2 \cdot \nabla [P_{01}] + \xi_1 \cdot \nabla (\mathbf{B}_{01} \cdot \mathbf{B}_1 - \mathbf{B}_{01}^{\vee} \cdot \mathbf{B}_1^{\vee})$$

$$\left. - (1/2) \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} [P_{01}] \right]$$

$$+ \mathbf{B}_1 \cdot \mathbf{B}_2 - \mathbf{B}_1^{\vee} \cdot \mathbf{B}_2^{\vee} - \xi_2 \cdot \nabla [P_1] - \xi_1 \cdot \nabla [P_2]$$

$$- \xi_1 \xi_2 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} [P_{00}] - (1/2) \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} [P_1]$$

$$- (1/6) \xi_1 \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} [P_{00}].$$

In deriving (61)–(62), the equilibrium conditions (53) and definitions (A14)–(A18), (A20) have been used.

According to the derivation, the boundary conditions (61) must be satisfied on the undisplaced plasma surface (12). If there are no equilibrium sur-

\* It should be noted that  $\mathfrak{P}$  is a scalar quantity.

face currents, according to (59) the terms  $\xi_i \cdot \nabla [P_{00}]$  in (61) and the terms  $\xi_1 \cdot \nabla [P_{0i}]$ ,  $\xi_2 \cdot \nabla [P_{0i}]$  in (62) vanish.

The time dependence of  $\mathfrak{P}_2$  follows from (33), (41) and (48). Using the definition (A19), we obtain

$$\mathfrak{P}_2 = \tau_1 A_1(T) \tilde{\mathfrak{P}}_{21}(\mathbf{r}) + (A_1^2(T)/2) \tilde{\mathfrak{P}}_{22}(\mathbf{r}) \quad (63)$$

with

$$\tilde{\mathfrak{P}}_{21} = \mathbf{B}_{01} \cdot \tilde{\mathbf{B}}_1 - \mathbf{B}_{01}^y \cdot \mathbf{B}_1^y - \xi_1 \cdot \nabla [P_{01}], \quad (64)$$

$$\tilde{\mathfrak{P}}_{22} = \tilde{B}_1^2 - \tilde{B}_1^{y^2} - 2 \xi_1 \cdot \nabla [\tilde{P}_1]$$

$$- \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} [P_{00}].$$

For obtaining the timedependence of the third order quantity  $\mathfrak{P}_3$ , in addition to (33), (35), (41), (42), (48), (49), (A14)–(A22) we make use of (91) which follow from the second order boundary conditions among (61), (82) and (89) and get

$$\begin{aligned} \mathfrak{P}_3 = & (\tau_1 A_2(T) + \tau_2 A_1(T)) \tilde{\mathfrak{P}}_{21}(\mathbf{r}) \\ & + A_1(T) A_2(T) \tilde{\mathfrak{P}}_{22}(\mathbf{r}) + \tau_1^2 A_1(T) \tilde{\mathfrak{P}}_{31}(\mathbf{r}) \quad (65) \\ & + \tau_1^2 A_1(T) \tilde{\mathfrak{P}}_{32}(\mathbf{r}) + (A_1^3(T)/2) \tilde{\mathfrak{P}}_{33}(\mathbf{r}) \end{aligned}$$

with

$$\begin{aligned} \tilde{\mathfrak{P}}_{31} = & \mathbf{B}_{01} \cdot \tilde{\mathbf{B}}_{21} - \mathbf{B}_{01}^y \cdot \tilde{\mathbf{B}}_{21}^y + \mathbf{B}_{02} \cdot \tilde{\mathbf{B}}_1 - \mathbf{B}_{02}^y \cdot \tilde{\mathbf{B}}_1^y \\ & - \xi_1 \cdot \nabla [P_{02}] - \xi_{21} \cdot \nabla [P_{01}] \end{aligned}$$

$$\tilde{\mathfrak{P}}_{32} = \tilde{\mathbf{B}}_1 \cdot \tilde{\mathbf{B}}_{21} - \tilde{\mathbf{B}}_1^y \cdot \tilde{\mathbf{B}}_{21}^y - \xi_{21} \cdot \nabla [\tilde{P}_1] - \xi_1 \cdot \nabla [\tilde{P}_{21}]$$

$$- \xi_1 \xi_{21} : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} [P_{00}]$$

$$+ (1/2) \left[ \mathbf{B}_{01} \cdot \tilde{\mathbf{B}}_{22} - \mathbf{B}_{01}^y \cdot \tilde{\mathbf{B}}_{22}^y - \xi_{22} \cdot \nabla [P_{01}] \right.$$

$$+ 2 \xi_1 \cdot \nabla (\mathbf{B}_{01} \cdot \tilde{\mathbf{B}}_1 - \mathbf{B}_{01}^y \cdot \tilde{\mathbf{B}}_1^y) \quad (66)$$

$$\left. - \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} [P_{01}] \right]$$

$$\tilde{\mathfrak{P}}_{33} = \tilde{\mathbf{B}}_1 \cdot \tilde{\mathbf{B}}_{22} - \tilde{\mathbf{B}}_1^y \cdot \tilde{\mathbf{B}}_{22}^y - \xi_{22} \cdot \nabla [\tilde{P}_1] - \xi_1 \cdot \nabla [\tilde{P}_{22}]$$

$$- \xi_1 \xi_{22} : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} [P_{00}] - \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} [\tilde{P}_1]$$

$$- (1/3) \xi_1 \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} [P_{00}].$$

#### 4.2. Condition $\mathbf{n} \times \partial \mathfrak{A} / \partial t = -\mathbf{n} \cdot \xi \mathbf{B}^y$ on $S$

In reducing the electric boundary condition (7) we shall first determine the normal vector  $\mathbf{n}$  on the displaced plasma surface  $S$ .

$$\mathbf{N} = \partial \mathbf{r} / \partial u \times \partial \mathbf{r} / \partial v \quad (67)$$

with  $\mathbf{r}$  given by (14) is a vector field perpendicular on  $S$ . Inserting (25) and (41)–(43) we get

$$\begin{aligned} N_{\mathbf{e}_{00}+\xi} = & N_{00}(\mathbf{r}_{00}) + \varepsilon A_1(T) \tilde{N}_1(\mathbf{r}_{00}) \\ & + \varepsilon^2 [A_2(T) \tilde{N}_{20}(\mathbf{r}_{00}) \quad (68) \\ & + \tau_1 A_1(T) \tilde{N}_{21}(\mathbf{r}_{00}) + (A_1^2(T)/2) \tilde{N}_{22}(\mathbf{r}_{00})], \end{aligned}$$

where

$$N_{00} = \partial \mathbf{r}_{00} / \partial u \times \partial \mathbf{r}_{00} / \partial v,$$

$$\tilde{N}_1 = \frac{\partial \mathbf{r}_{00}}{\partial u} \times \left( \frac{\partial \mathbf{r}_{00}}{\partial v} \cdot \nabla \tilde{\xi}_1 \right) + \left( \frac{\partial \mathbf{r}_{00}}{\partial u} \cdot \nabla \tilde{\xi}_1 \right) \times \frac{\partial \mathbf{r}_{00}}{\partial v},$$

$$\begin{aligned} \tilde{N}_{20} = & \frac{\partial \mathbf{r}_{00}}{\partial u} \times \left( \frac{\partial \mathbf{r}_{00}}{\partial v} \cdot \nabla \tilde{\xi}_{20} \right) + \left( \frac{\partial \mathbf{r}_{00}}{\partial u} \cdot \nabla \tilde{\xi}_{20} \right) \times \frac{\partial \mathbf{r}_{00}}{\partial v}, \quad (69) \\ \tilde{N}_{21} = & \frac{\partial \mathbf{r}_{00}}{\partial u} \times \left( \frac{\partial \mathbf{r}_{00}}{\partial v} \cdot \nabla \tilde{\xi}_{21} \right) + \left( \frac{\partial \mathbf{r}_{00}}{\partial u} \cdot \nabla \tilde{\xi}_{21} \right) \times \frac{\partial \mathbf{r}_{00}}{\partial v}, \end{aligned}$$

$$\begin{aligned} \tilde{N}_{22} = & \frac{\partial \mathbf{r}_{00}}{\partial u} \times \left( \frac{\partial \mathbf{r}_{00}}{\partial v} \cdot \nabla \tilde{\xi}_{22} \right) + \left( \frac{\partial \mathbf{r}_{00}}{\partial u} \cdot \nabla \tilde{\xi}_{22} \right) \times \frac{\partial \mathbf{r}_{00}}{\partial v} \\ & + 2 \left( \frac{\partial \mathbf{r}_{00}}{\partial u} \cdot \nabla \tilde{\xi}_1 \right) \times \left( \frac{\partial \mathbf{r}_{00}}{\partial v} \cdot \nabla \tilde{\xi}_1 \right). \end{aligned}$$

The normal vector  $\mathbf{n}$  can be expressed through  $\mathbf{N}$  by

$$\mathbf{n} = \mathbf{N} / \sqrt{N^2}. \quad (70)$$

Inserting (68) in (70), we obtain

$$\mathbf{n}|_{\mathbf{r}_{00}+\xi} = \mathbf{n}_{00} + \varepsilon \mathbf{n}_1 + \varepsilon^2 \mathbf{n}_2 + \dots \quad (71)$$

with

$$\mathbf{n}_{00} = \mathbf{N}_{00} / N_{00},$$

$$\mathbf{n}_1 = A_1(T) \tilde{\mathbf{n}}_1(\mathbf{r}), \quad \tilde{\mathbf{n}}_1 = (\tilde{N}_1 - \mathbf{n}_{00} \cdot \tilde{N}_1 \mathbf{n}_{00}) / N_{00},$$

$$\begin{aligned} \mathbf{n}_2 = & A_2(T) \tilde{\mathbf{n}}_{20}(\mathbf{r}_{00}) + \tau_1 A_1(T) \tilde{\mathbf{n}}_{21}(\mathbf{r}_{00}) \\ & + (A_1^2(T)/2) \tilde{\mathbf{n}}_{22}(\mathbf{r}_{00}), \quad (72) \end{aligned}$$

$$\tilde{\mathbf{n}}_{20} = (\tilde{N}_{20} - \mathbf{n}_{00} \cdot \tilde{N}_{20} \mathbf{n}_{00}) / N_{00},$$

$$\tilde{\mathbf{n}}_{21} = (\tilde{N}_{21} - \mathbf{n}_{00} \cdot \tilde{N}_{21} \mathbf{n}_{00}) / N_{00},$$

$$\tilde{\mathbf{n}}_{22} = (\tilde{N}_{22} - \mathbf{n}_{00} \cdot \tilde{N}_{22} \mathbf{n}_{00}) / N_{00} - 2 \mathbf{n}_{00} \cdot \frac{\tilde{N}_1}{N_{00}} \frac{\tilde{N}_1}{N_{00}}$$

$$+ \left[ 3 \left( \mathbf{n}_{00} \cdot \frac{\tilde{N}_1}{N_{00}} \right)^2 - \frac{\tilde{N}_1^2}{N_{00}^2} \right] \mathbf{n}_{00}.$$

Note that the right hand side of (71) is evaluated on the undisplaced plasmasurface (12).

For other representations of the displaced plasmasurface again (71) would be obtained, only  $\tilde{\mathbf{n}}_1$  etc. would have to be calculated differently.

Let us next evaluate

$$\partial \mathfrak{A} / \partial t = \sqrt{|\tau|} \left( \varepsilon \partial \mathfrak{A}_1 / \partial T + \varepsilon^2 \partial \mathfrak{A}_2 / \partial T + \varepsilon^3 \partial \mathfrak{A}_3 / \partial T + \dots \right)$$

on the displaced plasma surface. Setting now  $f_{00} \rightarrow 0$  and  $f_i \rightarrow \sqrt{|\tau|} \partial \mathfrak{A}_i / \partial T_i$  we may again use the general result (51) and obtain

$$\begin{aligned} \partial \mathfrak{A} / \partial t|_{r_0+\xi} = \sqrt{|\tau|} \left\{ \varepsilon \partial \mathfrak{A}_1 / \partial T \right. \\ + \varepsilon^2 (\partial \mathfrak{A}_2 / \partial T + \xi_1 \cdot \nabla \partial \mathfrak{A}_1 / \partial T) \\ + \varepsilon^3 \left( \partial \mathfrak{A}_3 / \partial T + \xi_1 \cdot \nabla \partial \mathfrak{A}_2 / \partial T \right. \\ + \xi_2 \cdot \nabla \partial \mathfrak{A}_1 / \partial T \\ \left. + (1/2) \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \partial \mathfrak{A}_1 / \partial T \right) + \dots \left. \right\}. \end{aligned}$$

From (13), with (22) and (25) we obtain

$$v|_{r_0+\xi} = \sqrt{|\tau|} \left( \varepsilon \partial \xi_1 / \partial T + \varepsilon^2 \partial \xi_2 / \partial T + \varepsilon^3 \partial \xi_3 / \partial T + \dots \right), \quad (74)$$

where again the right hand side is evaluated on  $S_{00}$ .

Finally, setting  $f_{00} \rightarrow B_{00}^v$ ,  $f_1 \rightarrow \tau_1 B_{01}^v + B_1^v$  etc., from (45) and (51) we obtain

$$\begin{aligned} B^v|_{r_0+\xi} = B_{00}^v + \varepsilon (\tau_1 B_{01}^v + B_1^v + \xi_1 \cdot \nabla B_{00}^v) \\ + \varepsilon^2 \left( \tau_2 B_{01}^v + \tau_1^2 B_{02}^v + B_2^v + \xi_2 \cdot \nabla B_{00}^v + \tau_1 \xi_1 \cdot \nabla B_{01}^v \right. \\ \left. + \xi_1 \cdot \nabla B_1^v + (1/2) \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} B_{00}^v \right) + \dots \quad (75) \end{aligned}$$

Inserting (71) and (73)–(75) in (7), the following conditions are obtained:

$$n_{00} \times \partial \mathfrak{A}_i / \partial T = -n_{00} \cdot \partial \xi_i / \partial T B_{00}^v - \partial \mathfrak{E}_i / \partial T, \quad i = 1, 2, 3, \dots, \quad (76)$$

where

$$\mathfrak{E}_1 = 0,$$

$$\begin{aligned} \partial \mathfrak{E}_2 / \partial T = n_{00} \times (\xi_1 \cdot \nabla \partial \mathfrak{A}_1 / \partial T) + n_1 \times \partial \mathfrak{A}_1 / \partial T \\ + n_1 \cdot \partial \xi_1 / \partial T B_{00}^v \\ + n_{00} \cdot \partial \xi_1 / \partial T (\tau_1 B_{01}^v + B_1^v + \xi_1 \cdot \nabla B_{00}^v), \quad (77) \end{aligned}$$

$$\begin{aligned} \partial \mathfrak{E}_3 / \partial T = n_{00} \times \left( \xi_1 \cdot \nabla \partial \mathfrak{A}_2 / \partial T + \xi_2 \cdot \nabla \partial \mathfrak{A}_1 / \partial T \right. \\ \left. + (1/2) \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \partial \mathfrak{A}_1 / \partial T \right) \\ + n_1 \times (\partial \mathfrak{A}_2 / \partial T + \xi_1 \cdot \nabla \partial \mathfrak{A}_1 / \partial T) \end{aligned}$$

$$\begin{aligned} + n_2 \times \partial \mathfrak{A}_1 / \partial T \\ + (n_1 \cdot \partial \xi_2 / \partial T + n_2 \cdot \partial \xi_1 / \partial T) B_{00}^v \\ + (n_{00} \cdot \partial \xi_2 / \partial T + n_1 \cdot \partial \xi_1 / \partial T) \\ (\tau_1 B_{01}^v + B_1^v + \xi_1 \cdot \nabla B_{00}^v) \\ + n_{00} \cdot \partial \xi_1 / \partial T \left( \tau_2 B_{01}^v + \tau_1^2 B_{02}^v + B_2^v \right. \\ \left. + \xi_2 \cdot \nabla B_{00}^v + \tau_1 \xi_1 \cdot \nabla B_{01}^v + \xi_1 \cdot \nabla B_1^v \right. \\ \left. + (1/2) \xi_1 \xi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} B_{00}^v \right). \end{aligned}$$

The fact that the vector fields  $\partial \mathfrak{E}_2 / \partial T$  and  $\partial \mathfrak{E}_3 / \partial T$  defined in (77) are indeed derivatives with respect to  $T$  can be seen by inserting the explicit time dependence of all quantities given in (41), (42), (48), (49) and (72) into the right hand sides. For example, the first term in  $\partial \mathfrak{E}_2 / \partial T$  is

$$\begin{aligned} n_{00} \times (\xi_1 \cdot \nabla \partial \mathfrak{A}_1 / \partial T) = A_1 \dot{A}_1 n_{00} \times (\tilde{\xi}_1 \cdot \nabla \tilde{\mathfrak{A}}_1) \\ = \frac{\partial}{\partial T} [(A_1^2/2) n_{00} \times (\tilde{\xi}_1 \cdot \nabla \tilde{A}_1)]. \end{aligned}$$

Another typical term contained in  $\partial \mathfrak{A}_3 / \partial T$  is

$$\begin{aligned} n_{00} \times (\xi_1 \cdot \nabla \partial \mathfrak{A}_2 / \partial T + \xi_2 \cdot \nabla \partial \mathfrak{A}_1 / \partial T) \\ = n_{00} \times [A_1 \dot{A}_2 \tilde{\xi}_1 \cdot \nabla \tilde{\mathfrak{A}}_{20} + \dot{A}_1 A_2 \tilde{\xi}_{20} \cdot \nabla \tilde{\mathfrak{A}}_1 \\ + \tau_1 A_1 \dot{A}_1 (\tilde{\xi}_1 \cdot \nabla \tilde{\mathfrak{A}}_{21} + \tilde{\xi}_{21} \cdot \nabla \tilde{\mathfrak{A}}_1) \\ + A_1 (A_1^2/2) \tilde{\xi}_1 \cdot \nabla \tilde{\mathfrak{A}}_{22} + (A_1^2/2) \dot{A}_1 \tilde{\xi}_{22} \cdot \nabla \tilde{\mathfrak{A}}_1] \\ = \frac{\partial}{\partial T} \{ n_{00} \times [a_{12} \tilde{\xi}_{20} \cdot \nabla \tilde{\mathfrak{A}}_1 + a_{21} \tilde{\xi}_1 \cdot \nabla \tilde{\mathfrak{A}}_{20} \\ + \tau_1 (A_1^2/2) (\tilde{\xi}_1 \cdot \nabla \tilde{\mathfrak{A}}_{21} + \tilde{\xi}_{21} \cdot \nabla \tilde{\mathfrak{A}}_1) \\ + (A_1^3/3) \tilde{\xi}_1 \cdot \nabla \tilde{\mathfrak{A}}_{22} + (A_1^3/6) \tilde{\xi}_{22} \cdot \nabla \tilde{\mathfrak{A}}_1] \}. \end{aligned}$$

If similar calculations are carried out for all other terms, and, if in evaluating the third order quantity  $\partial \mathfrak{A}_3 / \partial T$ , the second order results (91)–(94) are used, we finally obtain

$$\mathfrak{E}_2 = \tau_1 A_1(T) \tilde{\mathfrak{E}}_{21}(\mathbf{r}) + (A_1^2(T)/2) \tilde{\mathfrak{E}}_{22}(\mathbf{r}) \quad (78)$$

with

$$\begin{aligned} \tilde{\mathfrak{E}}_{21} = n_{00} \cdot \tilde{\xi}_1 B_{01}^v, \\ \tilde{\mathfrak{E}}_{22} = n_{00} \times (\tilde{\xi}_1 \cdot \nabla \tilde{\mathfrak{A}}_1) + \tilde{n}_1 \times \tilde{\mathfrak{A}}_1 + \tilde{n}_1 \cdot \tilde{\xi}_1 B_{00}^v \\ + n_{00} \cdot \tilde{\xi}_1 (\tilde{B}_1^v + \tilde{\xi}_1 \cdot \nabla B_{00}^v), \quad (79) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{E}_3 = (\tau_1 A_2(T) + \tau_2 A_1(T)) \tilde{\mathfrak{E}}_{21}(\mathbf{r}) \\ + A_1(T) A_2(T) \tilde{\mathfrak{E}}_{22}(\mathbf{r}) + \tau_1^2 A_1(T) \tilde{\mathfrak{E}}_{31}(\mathbf{r}) \\ + \tau_1 (A_1^2(T)/2) \tilde{\mathfrak{E}}_{32}(\mathbf{r}) + (A_1^3(T)/3) \tilde{\mathfrak{E}}_{33}(\mathbf{r}) \quad (80) \end{aligned}$$

with

$$\begin{aligned}\tilde{\mathfrak{C}}_{31} &= \mathbf{n}_{00} \cdot \tilde{\xi}_{21} B_{01}^v + \mathbf{n}_{00} \cdot \tilde{\xi}_1 B_{02}^v, \\ \tilde{\mathfrak{C}}_{32} &= \mathbf{n}_{00} \times (\tilde{\xi}_1 \cdot \nabla \tilde{\mathfrak{A}}_{21} + \tilde{\xi}_{21} \cdot \nabla \tilde{\mathfrak{A}}_1) + \tilde{\mathbf{n}}_1 \times \tilde{\mathfrak{A}}_{21} + \tilde{\mathbf{n}}_{21} \times \tilde{\mathfrak{A}}_1 \\ &\quad + (\tilde{\mathbf{n}}_1 \cdot \tilde{\xi}_{21} + \tilde{\mathbf{n}}_{21} \cdot \tilde{\xi}_1) B_{00}^v + \mathbf{n}_{00} \cdot \tilde{\xi}_{21} (\tilde{B}_1^v + \tilde{\xi}_1 \cdot \nabla B_{00}^v) \\ &\quad + \mathbf{n}_{00} \cdot \tilde{\xi}_1 (\tilde{B}_{21}^v + \tilde{\xi}_{21} \cdot \nabla B_{00}^v) \\ &\quad + (\mathbf{n}_{00} \cdot \tilde{\xi}_{22} + \tilde{\mathbf{n}}_1 \cdot \tilde{\xi}_1) B_{01}^v + (\mathbf{n}_{00} \cdot \tilde{\xi}_1) \tilde{\xi}_1 \cdot \nabla B_{01}^v, \quad (81)\end{aligned}$$

$$\begin{aligned}\tilde{\mathfrak{C}}_{33} &= \mathbf{n}_{00} \times \left( \tilde{\xi}_1 \cdot \nabla \tilde{\mathfrak{A}}_{22} + (1/2) \tilde{\xi}_{22} \cdot \nabla \tilde{\mathfrak{A}}_1 \right. \\ &\quad \left. + (1/2) \tilde{\xi}_1 \tilde{\xi}_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \tilde{\mathfrak{A}}_1 \right) \\ &\quad + \tilde{\mathbf{n}}_1 \times \tilde{\mathfrak{A}}_{22} + (1/2) \tilde{\mathbf{n}}_{22} \times \tilde{\mathfrak{A}}_1 + \tilde{\mathbf{n}}_1 \times (\tilde{\xi}_1 \cdot \nabla \tilde{\mathfrak{A}}_1) \\ &\quad + (\tilde{\mathbf{n}}_1 \cdot \tilde{\xi}_{22} + (1/2) \tilde{\mathbf{n}}_{22} \cdot \tilde{\xi}_1) B_{00}^v \\ &\quad + \mathbf{n}_{00} \cdot \tilde{\xi}_{22} (\tilde{B}_1^v + \tilde{\xi}_1 \cdot \nabla B_{00}^v) \\ &\quad + (1/2) \mathbf{n}_{00} \cdot \tilde{\xi}_1 (\tilde{B}_{22}^v + \tilde{\xi}_{22} \cdot \nabla B_{00}^v) \\ &\quad + \tilde{\mathbf{n}}_1 \cdot \tilde{\xi}_1 (\tilde{B}_1^v + \tilde{\xi}_1 \cdot \nabla B_{00}^v) + (\mathbf{n}_{00} \cdot \tilde{\xi}_1) \tilde{\xi}_1 \cdot \nabla \tilde{B}_{01}^v \\ &\quad + (1/2) (\mathbf{n}_{00} \cdot \tilde{\xi}_1) \tilde{\xi}_1 \tilde{\xi}_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} B_{00}^v.\end{aligned}$$

Since  $\mathbf{n}_{00}$  and  $B_{00}^v$  are timeindependent, integration of (76) with respect to  $T$  yields

$$\mathbf{n}_{00} \times \tilde{\mathfrak{A}}_i = -\mathbf{n}_{00} \cdot \tilde{\xi}_i B_{00}^v - \mathfrak{C}_i, \quad i = 1, 2, 3, \dots$$

with  $\mathfrak{C}_1 = 0$  (first of (77)) and  $\mathfrak{C}_2, \mathfrak{C}_3$  given by (78)–(81). Integration functions  $g_i(\mathbf{r})$  which could be added on the right hand sides of (82) are set zero such that  $\tilde{\mathfrak{A}}_i = 0$  if all perturbational quantities on the right hand are zero. Note that according to the derivation the reduced boundary conditions (82) are again evaluated on the undisplaced plasma surface.

#### 4.3. Condition $\mathbf{n} \times \tilde{\mathfrak{A}} = 0$ on $\mathfrak{C}_0$

The treatment of the electric boundary condition on the conducting wall  $\mathfrak{C}_0$  is slightly different from that on  $S$  since, on the one hand,  $\mathfrak{C}_0$  is not shifted during the plasma motion while, on the other, we admit a dependence of  $\mathfrak{C}_0$  on the driving parameter. Let

$$\begin{aligned}\mathbf{r} &= \mathbf{r}_0(u, v, \lambda) = \mathbf{r}_{00}(u, v) + \tau \mathbf{r}_{01}(u, v) \\ &\quad + \tau^2 \mathbf{r}_{02}(u, v) + \dots\end{aligned} \quad (83)$$

be a parameter representation of  $\mathfrak{C}_0$ . Again

$$\begin{aligned}N &= \frac{\partial \mathbf{r}_0}{\partial u} \times \frac{\partial \mathbf{r}_0}{\partial v} = N_{00} + \varepsilon \tau_1 N_{01} \\ &\quad + \varepsilon^2 (\tau_2 N_{01} + \tau_1^2 N_{02}) + \dots\end{aligned} \quad (84)$$

with

$$\begin{aligned}N_{00} &= \frac{\partial \mathbf{r}_{00}}{\partial u} \times \frac{\partial \mathbf{r}_{00}}{\partial v}, \\ N_{01} &= \frac{\partial \mathbf{r}_{00}}{\partial u} \times \frac{\partial \mathbf{r}_{01}}{\partial v} + \frac{\partial \mathbf{r}_{01}}{\partial u} \times \frac{\partial \mathbf{r}_{00}}{\partial v}, \\ N_{02} &= \frac{\partial \mathbf{r}_{00}}{\partial u} \times \frac{\partial \mathbf{r}_{02}}{\partial v} + \frac{\partial \mathbf{r}_{02}}{\partial u} \times \frac{\partial \mathbf{r}_{00}}{\partial v} + \frac{\partial \mathbf{r}_{01}}{\partial u} \times \frac{\partial \mathbf{r}_{01}}{\partial v}\end{aligned} \quad (85)$$

is a vectorfield perpendicular on  $\mathfrak{C}_0$ , while from (70) and (84) we obtain now

$$\mathbf{n} = \mathbf{n}_{00} + \varepsilon \tau_1 \mathbf{n}_{01} + \varepsilon^2 (\tau_2 \mathbf{n}_{01} + \tau_1^2 \mathbf{n}_{02}) + \dots \quad (86)$$

with

$$\begin{aligned}\mathbf{n}_{00} &= N_{00}/N_{00}, \\ \mathbf{n}_{01} &= (N_{01} - \mathbf{n}_{00} \cdot N_{01} \mathbf{n}_{00})/N_{00}, \\ \mathbf{n}_{02} &= (N_{02} - \mathbf{n}_{00} \cdot N_{02} \mathbf{n}_{00})/N_{00} - \mathbf{n}_{00} \cdot N_{01} N_{01}/N_{00}^2 \\ &\quad + (1/2) [3 (\mathbf{n}_{00} \cdot N_{01}/N_{00})^2 - N_{01}^2/N_{00}^2] \mathbf{n}_{00}\end{aligned} \quad (87)$$

for the normal vectorfield on  $\mathfrak{C}_0$ . Note that  $\mathbf{n}$  is the timeindependent vectorfield on the surface  $\mathbf{r}_0(u, v, \lambda)$  while  $\mathbf{n}_{00}, \mathbf{n}_{01}, \mathbf{n}_{02}, \dots$  are evaluated on the marginal position  $\mathbf{r}_{00}(u, v) = \mathbf{r}_0(u, v, \lambda_0)$  of  $\mathfrak{C}_0$ .

Condition (16) must be evaluated on  $\mathbf{r} = \mathbf{r}_0(u, v, \lambda)$ , and hence we need

$$\begin{aligned}\tilde{\mathfrak{A}}(\mathbf{r}_0, t) &= \tilde{\mathfrak{A}}(\mathbf{r}_{00} + \tau \mathbf{r}_{01} + \tau^2 \mathbf{r}_{02} + \dots, t) \\ &= \left[ \tilde{\mathfrak{A}} + \tau \mathbf{r}_{01} + \tau^2 \mathbf{r}_{02} \cdot \nabla \tilde{\mathfrak{A}} \right. \\ &\quad \left. + (1/2) \tau^2 \mathbf{r}_{01} \mathbf{r}_{01} : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \tilde{\mathfrak{A}} \right] \Big|_{\mathbf{r}_{00}} \\ &= \left[ \varepsilon \tilde{\mathfrak{A}}_1 + \varepsilon^2 (\tilde{\mathfrak{A}}_2 + \tau_1 \mathbf{r}_{01} \cdot \nabla \tilde{\mathfrak{A}}_1) \right. \\ &\quad \left. + \varepsilon^3 (\tilde{\mathfrak{A}}_3 + \tau_2 \mathbf{r}_{01} \cdot \nabla \tilde{\mathfrak{A}}_1 + \tau_1^2 \mathbf{r}_{02} \cdot \nabla \tilde{\mathfrak{A}}_1 \right. \\ &\quad \left. + \tau_1 \mathbf{r}_{01} \cdot \nabla \tilde{\mathfrak{A}}_2 \right. \\ &\quad \left. + (1/2) \tau_1^2 \mathbf{r}_{01} \mathbf{r}_{01} : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \tilde{\mathfrak{A}}_1) \right] \Big|_{\mathbf{r}_{00}} + \dots\end{aligned} \quad (88)$$

With (86) and (88), from (16) we get the boundary conditions

$$\mathbf{n}_{00} \times \tilde{\mathfrak{A}}_i = -\mathfrak{C}_i, \quad i = 1, 2, 3, \dots, \quad (89)$$

where

$$\begin{aligned}\mathfrak{C}_1 &= 0, \\ \mathfrak{C}_2 &= \tau_1 A_1(T) \tilde{\mathfrak{C}}_{21}(\mathbf{r}), \\ \tilde{\mathfrak{C}}_{21} &= \mathbf{n}_{00} \times \mathbf{r}_{01} \cdot \nabla \tilde{\mathfrak{A}}_1 + \mathbf{n}_{01} \times \tilde{\mathfrak{A}}_1,\end{aligned} \quad (90)$$



$$\begin{aligned}
\mathfrak{C}_3 &= (\tau_1 A_2(T) + \tau_2 A_1(T)) \tilde{\mathfrak{C}}_{21}(\mathbf{r}) \\
&\quad + \tau_1^2 A_1(T) \tilde{\mathfrak{C}}_{31}(\mathbf{r}) + \tau_1 (A_1^2(T)/2) \mathfrak{C}_{32}(\mathbf{r}), \\
\tilde{\mathfrak{C}}_{31} &= \mathbf{n}_{00} \times (\mathbf{r}_{01} \cdot \nabla \tilde{\mathfrak{A}}_{21}) + \mathbf{n}_{01} \times \tilde{\mathfrak{A}}_{21} \\
&\quad + \mathbf{n}_{00} \times \left( \mathbf{r}_{02} \cdot \nabla \tilde{\mathfrak{A}}_1 + (1/2) \mathbf{r}_{01} \mathbf{r}_{01} : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \tilde{\mathfrak{A}}_1 \right) \\
&\quad + \mathbf{n}_{01} \times (\mathbf{r}_{01} \cdot \nabla \tilde{\mathfrak{A}}_1) + \mathbf{n}_{02} \times \tilde{\mathfrak{A}}_1, \\
\tilde{\mathfrak{C}}_{32} &= \mathbf{n}_{00} \times (\mathbf{r}_{01} \cdot \nabla \tilde{\mathfrak{A}}_{22}) + \mathbf{n}_{01} \times \tilde{\mathfrak{A}}_{22}.
\end{aligned}$$

For calculating  $\mathfrak{C}_3$ , again use has been made of (91). According to the derivation, (89) must be evaluated on  $\mathfrak{C}_{00}$ . When the position of the external wall does not depend on  $\lambda$ , we have  $\mathfrak{C}_1 = \mathfrak{C}_2 = \mathfrak{C}_3 = 0$ .

#### 4.4. Proof of $\tilde{\xi}_{20} = \Phi_{20} = \Phi_1$ etc.

According to (27), (30), (47), and (49),  $\Phi_{20}$  and  $\tilde{\mathfrak{A}}_{20}$  satisfy the same equations as  $\Phi_1$  and  $\tilde{\mathfrak{A}}_1$ . Using (29), (35), (36), (42), and (49), it follows from (61) with (63), (82) with (78) and from (89) with (90) that they satisfy also the same boundary conditions as  $\Phi_1$  and  $\tilde{\mathfrak{A}}_1$ . Since, by assumption, we consider the transition of a nondegenerate eigenvalue from stable to unstable, there exists only one solution of the marginal mode eigenvalue problem and therefore we must have

$$\tilde{\xi}_{20} = \Phi_{20} = \Phi_1 \quad \text{a)} \quad \tilde{\mathfrak{A}}_{20} = \tilde{\mathfrak{A}}_1 \quad \text{b).} \quad (91)$$

From (34), (36) and (91) we get

$$\tilde{p}_{20} = \tilde{p}_1, \quad \tilde{\mathbf{B}}_{20} = \tilde{\mathbf{B}}_1, \quad \tilde{\mathbf{B}}_{20}^\vee = \tilde{\mathbf{B}}_1^\vee, \quad (92)$$

and from this, according to the definitions (A19) and (A21) there follows

$$[\tilde{P}_{20}] = [\tilde{P}_1]. \quad (93)$$

Finally, from (69) and (72) we get

$$\tilde{\mathbf{n}}_{20} = \tilde{\mathbf{n}}_1. \quad (94)$$

## 5. Amplitude Equations for the Marginal Mode

### 5.1. Derivation

Similarly as in the case of internal modes, solubility conditions for the equations of motion (28) and (31) yield nonlinear differential equations for the amplitudes  $A_1(T)$  and  $A_2(T)$  of the marginal mode  $\Phi_1 = \Phi_{20}$ . Such conditions are obtained by multiplying these equations with  $\Phi_1$  and integrating them over the unperturbed plasma volume. As was already mentioned in the introduction,  $\mathbf{F}_{00}$  is not generally selfadjoint but this property is restricted to the subspace of functions which satisfy the boundary conditions of linear stability theory ((61), (82), and (89) for  $i = 1$ ). More specifically, using the definition (A2) we have in general

$$(\Phi_1, \mathbf{F}_{00}(\Phi_i)) \neq (\Phi_i, \mathbf{F}_{00}(\Phi_1)) = 0, \quad i = 2, 3 \quad (95)$$

and thus the terms  $(\Phi_1, \mathbf{F}_{00}(\Phi_i))$  containing the highest order contributions do not vanish as for internal modes. However, these contributions can now be eliminated with proper integration by parts.

Thus, like in linear stability theory one obtains

$$-(\Phi_1, \mathbf{F}_{00}(\Phi_i)) = 2 \delta^2 W_{00}^{\text{Pl}}(\Phi_1, \Phi_i) + \int_{S_{00}} \mathbf{n}_{00} \cdot \Phi_1 [\mathbf{p}_{10}(\Phi_i) + \mathbf{B}_{00} \cdot \mathbf{B}_{10}(\Phi_i)] dS \quad (96)$$

and

$$0 = -(\Phi_i, \mathbf{F}_{00}(\Phi_1)) = 2 \delta^2 W_{00}^{\text{Pl}}(\Phi_i, \Phi_1) + \int_{S_{00}} \mathbf{n}_{00} \cdot \Phi_i (p_1 + \mathbf{B}_{00} \cdot \mathbf{B}_1) dS, \quad (97)$$

where (33), (34), (A5), (A6), and (A23) are used.  $\delta^2 W_{00}^{\text{Pl}}(\Phi_1, \Phi_i)$  can be brought into a symmetric form [3], and hence

$$\delta^2 W_{00}^{\text{Pl}}(\Phi_i, \Phi_1) = \delta^2 W_{00}^{\text{Pl}}(\Phi_1, \Phi_i). \quad (98)$$

With this,  $\Phi_1 = \xi_1$  (see (38)) and the boundary condition (61), from (96)–(97) we obtain

$$\begin{aligned}
-(\Phi_1, \mathbf{F}_{00}(\Phi_i)) &= \int_{S_{00}} \mathbf{n}_{00} \cdot \xi_1 [\mathbf{p}_{10}(\Phi_i) - p_i + \mathbf{B}_{00} \cdot (\mathbf{B}_{10}(\Phi_i) - \mathbf{B}_i) + \mathbf{B}_{00}^\vee \cdot \mathbf{B}_i^\vee + \xi_i \cdot \nabla [P_{00}] - \mathfrak{P}_i] dS \\
&\quad - \int_{S_{00}} \mathbf{n}_{00} \cdot \xi_i (\mathbf{B}_{00}^\vee \cdot \mathbf{B}_i^\vee + \xi_1 \cdot \nabla [P_{00}]) dS - \int_{S_{00}} \mathbf{n}_{00} \cdot (\Phi_i - \xi_i) (\mathbf{B}_{00}^\vee \cdot \mathbf{B}_1^\vee + \xi_1 \cdot \nabla [P_{00}]) dS.
\end{aligned} \quad (99)$$

Since  $[P_{00}] \equiv 0$  on  $S_{00}$  (see (53)),  $\nabla [P_{00}]$  is perpendicular on  $S_{00}$  and thus

$$(\mathbf{n}_{00} \cdot \xi_1) \xi_i \cdot \nabla [P_{00}] - (\mathbf{n}_{00} \cdot \xi_i) \xi_1 \cdot \nabla [P_{00}] = 0. \quad (100)$$

Other integrations by parts well known from linear stability theory yield

$$\int_{I_{ac}^{00}} \mathbf{B}_i^\vee \cdot \mathbf{B}_i^\vee d\tau = \int_{\mathfrak{C}_{00}} (\mathfrak{A}_i \times \mathbf{B}_i^\vee) \cdot \mathbf{n}_{00} dS - \int_{S_{00}} (\mathfrak{A}_i \times \mathbf{B}_i^\vee) \cdot \mathbf{n}_{00} dS = \int_{\mathfrak{C}_{00}} (\mathfrak{A}_1 \times \mathbf{B}_i^\vee) \cdot \mathbf{n}_{00} dS - \int_{S_{00}} (\mathfrak{A}_1 \times \mathbf{B}_i^\vee) \cdot \mathbf{n}_{00} dS, \quad (101)$$

where the volume integration on the left hand side extends over the vacuum region between the undisplaced plasma surface  $S_{00}$  and the marginal position  $\mathfrak{C}_{00}$  of the external wall. If the boundary conditions (82) and (89) are inserted in (101), we obtain

$$\int_{S_{00}} [(\mathbf{n}_{00} \cdot \xi_1) \mathbf{B}_{00}^\vee \cdot \mathbf{B}_i^\vee - (\mathbf{n}_{00} \cdot \xi_i) \mathbf{B}_{00}^\vee \cdot \mathbf{B}_1^\vee] dS = \int_{S_{00}} \mathfrak{C}_i \cdot \mathbf{B}_i^\vee dS - \int_{\mathfrak{C}_{00}} \mathfrak{C}_i \cdot \mathbf{B}_i^\vee dS. \quad (102)$$

With (100) and (102) we finally obtain from (99)

$$\begin{aligned} -(\Phi_1, \mathbf{F}_{00}(\Phi_i)) &= \int_{S_{00}} \{\mathbf{n}_{00} \cdot \xi_1 [\mathbf{p}_{10}(\Phi_i) - p_i + \mathbf{B}_{00} \cdot (\mathbf{B}_{10}(\Phi_i) - \mathbf{B}_i) - \mathfrak{P}_i] \\ &\quad - \mathbf{n}_{00} \cdot (\Phi_i - \xi_i) [\mathbf{B}_{00}^\vee \cdot \mathbf{B}_i^\vee + \xi_1 \cdot \nabla [P_{00}]] + \mathfrak{C}_i \cdot \mathbf{B}_i^\vee\} dS - \int_{\mathfrak{C}_{00}} \mathfrak{C}_i \cdot \mathbf{B}_i^\vee dS. \end{aligned} \quad (103)$$

Using (26), (29), (35), (36), (38), (41), (48), (63), (78), (90), and (91), we get from (103) for  $i = 2$

$$\begin{aligned} A_1^{-1}(\Phi_1, \mathbf{F}_{00}(\Phi_2)) &= \tau_1 A_1 \left\{ \int_{S_{00}} [\mathbf{n}_{00} \cdot \Phi_1 (\mathbf{p}_{11}(\Phi_1) + \mathbf{B}_{00} \cdot \mathbf{B}_{11}(\Phi_1) + \mathfrak{P}_{21}) - \tilde{\mathfrak{C}}_{21} \cdot \tilde{\mathbf{B}}_1^\vee] dS + \int_{\mathfrak{C}_{00}} \mathfrak{C}_{21} \cdot \tilde{\mathbf{B}}_1^\vee dS \right\} \\ &\quad + (A_1^2/2) \int_{S_{00}} \{\mathbf{n}_{00} \cdot \Phi_1 [\mathbf{p}_1(\Phi_1, \mathbf{p}_{10}(\Phi_1)) + \mathbf{B}_{00} \cdot \mathbf{B}_1(\Phi_1, \mathbf{B}_{10}(\Phi_1)) + \mathfrak{P}_{22}] \\ &\quad - \mathbf{n}_{00} \cdot (\Phi_1 \cdot \nabla) \Phi_1 (\mathbf{B}_{00}^\vee \cdot \tilde{\mathbf{B}}_1^\vee + \Phi_1 \cdot \nabla [P_{00}]) - \tilde{\mathfrak{C}}_{22} \cdot \tilde{\mathbf{B}}_1^\vee\} dS. \end{aligned} \quad (104)$$

With this and (26), from (28) we obtain the amplitude equation

$$\tau_1 (\ddot{A}_1 - \gamma_T^2 A_1) = (3/2) \delta_T A_1^2, \quad (105)$$

where

$$\gamma_T^2 = \mathcal{Q}_{00}^{-1} \left\{ (\Phi_1, \mathbf{F}_{01}(\Phi_1)) + \int_{S_{00}} \{\mathbf{n}_{00} \cdot \Phi_1 (\mathbf{p}_{11}(\Phi_1) + \mathbf{B}_{00} \cdot \mathbf{B}_{11}(\Phi_1) + \mathfrak{P}_{21}) - \tilde{\mathfrak{C}}_{21} \cdot \tilde{\mathbf{B}}_1^\vee\} dS + \int_{\mathfrak{C}_{00}} \tilde{\mathfrak{C}}_{21} \cdot \tilde{\mathbf{B}}_1^\vee dS \right\}, \quad (106)$$

$$\begin{aligned} \delta_T &= (3 \mathcal{Q}_{00})^{-1} \left\{ (\Phi_1, \mathbf{G}_{00}(\Phi_1, \Phi_1)) + \int_{S_{00}} \{\mathbf{n}_{00} \cdot \Phi_1 (\mathbf{p}_1(\Phi_1, \mathbf{p}_{10}(\Phi_1)) + \mathbf{B}_{00} \cdot \mathbf{B}_1(\Phi_1, \mathbf{B}_{10}(\Phi_1)) + \mathfrak{P}_{22}) \right. \\ &\quad \left. - \mathbf{n}_{00} \cdot (\Phi_1 \cdot \nabla) \Phi_1 (\mathbf{B}_{00}^\vee \cdot \tilde{\mathbf{B}}_1^\vee + \Phi_1 \cdot \nabla [P_{00}]) - \tilde{\mathfrak{C}}_{22} \cdot \tilde{\mathbf{B}}_1^\vee\} dS \right\}, \end{aligned} \quad (107)$$

and where the normalization

$$(\Phi_1, \Phi_1) = 1 \quad (108)$$

is used.

Using (26), (37), (38), (41), (48), (65), (80), and (90), we get from (103) for  $i = 3$

$$\begin{aligned} A_1^{-1}(\Phi_1, \mathbf{F}_{00}(\Phi_3)) &= (\tau_1 A_2 + \tau_2 A_1) \left\{ \int_{S_{00}} \{\mathbf{n}_{00} \cdot \Phi_1 (\mathbf{p}_{11}(\Phi_1) + \mathbf{B}_{00} \cdot \mathbf{B}_{11}(\Phi_1) + \mathfrak{P}_{21}) \right. \\ &\quad \left. - \tilde{\mathfrak{C}}_{21} \cdot \tilde{\mathbf{B}}_1^\vee\} dS + \int_{\mathfrak{C}_{00}} \tilde{\mathfrak{C}}_{21} \cdot \tilde{\mathbf{B}}_1^\vee dS \right\} \\ &\quad + A_1 A_2 \int_{S_{00}} \{\mathbf{n}_{00} \cdot \Phi_1 (\mathbf{p}_1(\Phi_1, \mathbf{p}_{10}(\Phi_1)) + \mathbf{B}_{00} \cdot \mathbf{B}_1(\Phi_1, \mathbf{B}_{10}(\Phi_1)) + \mathfrak{P}_{22}) \\ &\quad - \mathbf{n}_{00} \cdot (\Phi_1 \cdot \nabla) \Phi_1 (\mathbf{B}_{00}^\vee \cdot \tilde{\mathbf{B}}_1^\vee + \Phi_1 \cdot \nabla [P_{00}]) - \tilde{\mathfrak{C}}_{22} \cdot \tilde{\mathbf{B}}_1^\vee\} dS \end{aligned} \quad (109)$$

$$\begin{aligned}
& + \tau_1^2 A_1 \left\{ \int_{S_{00}} \{ \mathbf{n}_{00} \cdot \Phi_1 [\mathbf{p}_{11}(\Phi_{21}) + \mathbf{p}_{12}(\Phi_1) + \mathbf{B}_{00} \cdot (\mathbf{B}_{11}(\Phi_{21}) + \mathbf{B}_{12}(\Phi_1)) + \tilde{\mathfrak{P}}_{31}] - \tilde{\mathfrak{C}}_{31} \cdot \tilde{\mathbf{B}}_1^\vee \} dS + \int_{\mathfrak{C}_{00}} \tilde{\mathfrak{C}}_{31} \cdot \tilde{\mathbf{B}}_1^\vee dS \right\} \\
& + \tau_1 (A_1^2/2) \left\{ \int_{S_{00}} \{ \mathbf{n}_{00} \cdot \Phi_1 [\mathbf{p}_{11}(\Phi_{22}) + \mathbf{p}_1(\Phi_1, \mathbf{p}_{10}(\Phi_{21})) + \mathbf{p}_1(\Phi_{21}, \mathbf{p}_{10}(\Phi_1)) \right. \\
& \quad + \mathbf{p}_1(\Phi_1, \mathbf{p}_{11}(\Phi_1)) + \mathbf{B}_{00} \cdot (\mathbf{B}_{11}(\Phi_{22}) + \mathbf{B}_1(\Phi_1, \mathbf{B}_{10}(\Phi_{21})) + \mathbf{B}_1(\Phi_{21}, \mathbf{B}_{10}(\Phi_1)) \\
& \quad + \mathbf{B}_1(\Phi_1, \mathbf{B}_{11}(\Phi_1))) + 2\tilde{\mathfrak{P}}_{32}] - \mathbf{n}_{00} \cdot (\Phi_{21} \cdot \nabla \Phi_1 + \Phi_1 \cdot \nabla \Phi_{21}) (B_{00}^\vee \cdot \tilde{\mathbf{B}}_1^\vee + \Phi_1 \cdot \nabla [P_{00}]) \\
& \quad \left. - \tilde{\mathfrak{C}}_{32} \cdot \tilde{\mathbf{B}}_1^\vee \} dS + \int_{\mathfrak{C}_{00}} \tilde{\mathfrak{C}}_{32} \cdot \tilde{\mathbf{B}}_1^\vee dS \right\} + (A_1^3/6) \int_{S_{00}} \left\{ \mathbf{n}_{00} \cdot \Phi_1 [\mathbf{p}_1(\Phi_1, \mathbf{p}_{10}(\Phi_{22})) + 2\mathbf{p}_1(\Phi_{22}, \mathbf{p}_{10}(\Phi_1)) \right. \\
& \quad + \mathbf{p}_1(\Phi_1, \mathbf{p}_1(\Phi_1, \mathbf{p}_{10}(\Phi_1))) + \mathbf{B}_{00} \cdot (\mathbf{B}_1(\Phi_1, \mathbf{B}_{10}(\Phi_{22})) + 2\mathbf{B}_1(\Phi_{22}, \mathbf{B}_{10}(\Phi_1)) \\
& \quad + \mathbf{B}_1(\Phi_1, \mathbf{B}_1(\Phi_1, \mathbf{B}_{10}(\Phi_1)))) + 3\tilde{\mathfrak{P}}_{33}] - \mathbf{n}_{00} \cdot \left( \Phi_{22} \cdot \nabla \Phi_1 + 2\Phi_1 \cdot \nabla \Phi_{22} \right. \\
& \quad \left. + (\Phi_1 \cdot \nabla \Phi_1) \cdot \nabla \Phi_1 + \Phi_1 \Phi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \Phi_1 \right) (B_{00}^\vee \cdot \tilde{\mathbf{B}}_1^\vee + \Phi_1 \cdot \nabla [P_{00}]) - 2\tilde{\mathfrak{C}}_{33} \cdot \tilde{\mathbf{B}}_1^\vee \} dS.
\end{aligned}$$

With this, (26), (32 b), (91 a), and (108) from (31) we obtain the amplitude equation

$$\begin{aligned}
& \tau_2 (\ddot{A}_1 - \gamma_T^2 A_1) + \tau_1 (\ddot{A}_2 - \gamma_T^2 A_2) + \tau_1 (\tau_1 a_1 \ddot{A}_1 + a_2 \ddot{A}_1 A_1 + b \dot{A}_1^2) \\
& = 3 \delta_T A_1 A_2 + \tau_1 (\tau_1 c_1 A_1 + c_2 A_1^2) + 2 \vartheta_T A_1^3,
\end{aligned} \tag{110}$$

where  $\gamma_T^2$  and  $\delta_T$  are given by (106) and (107) and where  $a_1, a_2, b, c_1, c_2$  and  $\vartheta_T$  are defined by

$$\begin{aligned}
a_1 &= (\Phi_1, \Phi_{21}), \quad a_2 = (\Phi_1, -\Phi_1 \cdot \nabla \Phi_1 + \Phi_{22}), \\
b &= (\Phi_1, \Phi_1 \cdot \nabla \Phi_1 + \Phi_{22}),
\end{aligned} \tag{111}$$

$$\begin{aligned}
c_1 &= \varrho_{00}^{-1} (\Phi_1, \mathbf{F}_{01}(\Phi_{21}) + \mathbf{F}_{02}(\Phi_1)) + \varrho_{00}^{-1} \left\{ \int_{S_{00}} \{ \mathbf{n}_{00} \cdot \Phi_1 [\mathbf{p}_{11}(\Phi_{21}) + \mathbf{p}_{12}(\Phi_1) + \mathbf{B}_{00} \cdot (\mathbf{B}_{11}(\Phi_{21}) \right. \\
& \quad \left. + \mathbf{B}_{12}(\Phi_1)) + \tilde{\mathfrak{P}}_{31}] - \tilde{\mathfrak{C}}_{31} \cdot \tilde{\mathbf{B}}_1^\vee \} dS + \int_{\mathfrak{C}_{00}} \tilde{\mathfrak{C}}_{31} \cdot \tilde{\mathbf{B}}_1^\vee dS \right\}, \\
c_2 &= (2\varrho_{00})^{-1} (\Phi_1, \mathbf{F}_{01}(\Phi_{22}) + \mathbf{G}_{01}(\Phi_1, \Phi_1) + \mathbf{G}_{00}(\Phi_1, \Phi_{21}) + \mathbf{G}_{00}(\Phi_{21}, \Phi_1))
\end{aligned}$$

$$\begin{aligned}
& + (2\varrho_{00})^{-1} \left\{ \int_{S_{00}} \mathbf{n}_{00} \cdot \Phi_1 [\mathbf{p}_{11}(\Phi_{22}) + \mathbf{p}_1(\Phi_1, \mathbf{p}_{10}(\Phi_{21})) + \mathbf{p}_1(\Phi_{21}, \mathbf{p}_{10}(\Phi_1)) + \mathbf{p}_1(\Phi_1, \mathbf{p}_{11}(\Phi_1)) \right. \\
& \quad + \mathbf{B}_{00} \cdot (\mathbf{B}_{11}(\Phi_{22}) + \mathbf{B}_1(\Phi_1, \mathbf{B}_{10}(\Phi_{21})) + \mathbf{B}_1(\Phi_{21}, \mathbf{B}_{10}(\Phi_1)) + \mathbf{B}_1(\Phi_1, \mathbf{B}_{11}(\Phi_1))) \\
& \quad + 2\tilde{\mathfrak{P}}_{32}] - \mathbf{n}_{00} \cdot (\Phi_{21} \cdot \nabla \Phi_1 + \Phi_1 \cdot \nabla \Phi_{21}) (B_{00}^\vee \cdot \tilde{\mathbf{B}}_1^\vee + \Phi_1 \cdot \nabla [P_{00}]) \\
& \quad \left. - \tilde{\mathfrak{C}}_{32} \cdot \tilde{\mathbf{B}}_1^\vee \} dS + \int_{\mathfrak{C}_{00}} \tilde{\mathfrak{C}}_{32} \cdot \tilde{\mathbf{B}}_1^\vee dS \right\},
\end{aligned}$$

$$\begin{aligned}
\vartheta_T &= (12\varrho_{00})^{-1} (\Phi_1, \mathbf{G}_{00}(\Phi_1, \Phi_{22}) + 2\mathbf{G}_{00}(\Phi_{22}, \Phi_1) + \mathbf{H}_{00}(\Phi_1, \Phi_1, \Phi_1)) \\
& + (12\varrho_{00})^{-1} \int_{S_{00}} \left\{ \mathbf{n}_{00} \cdot \Phi_1 [\mathbf{p}_1(\Phi_1, \mathbf{p}_{10}(\Phi_{22})) + 2\mathbf{p}_1(\Phi_{22}, \mathbf{p}_{10}(\Phi_1)) + \mathbf{p}_1(\Phi_1, \mathbf{p}_1(\Phi_1, \mathbf{p}_{10}(\Phi_1))) \right. \\
& \quad + \mathbf{B}_{00} \cdot (\mathbf{B}_1(\Phi_1, \mathbf{B}_{10}(\Phi_{22})) + 2\mathbf{B}_1(\Phi_{22}, \mathbf{B}_{10}(\Phi_1)) + \mathbf{B}_1(\Phi_1, \mathbf{B}_1(\Phi_1, \mathbf{B}_{10}(\Phi_1)))) + 3\tilde{\mathfrak{P}}_{33}] \\
& \quad - \mathbf{n}_{00} \cdot \left( \Phi_{22} \cdot \nabla \Phi_1 + 2\Phi_1 \cdot \nabla \Phi_{22} + (\Phi_1 \cdot \nabla \Phi_1) \cdot \nabla \Phi_1 + \Phi_1 \Phi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \Phi_1 \right) \\
& \quad \left. (B_{00}^\vee \cdot \tilde{\mathbf{B}}_1^\vee + \Phi_1 \cdot \nabla [P_{00}]) - 2\tilde{\mathfrak{C}}_{33} \cdot \tilde{\mathbf{B}}_1^\vee \right\} dS.
\end{aligned} \tag{112}$$

The amplitude equations (105) and (110) are formally identical with those obtained for internal modes. These can be obtained as a special case of external modes for  $\mathbf{n}_{00} \cdot \Phi = 0$  and  $\mathfrak{A} = 0$ . Under these conditions, all surface integral contributions to the coefficients of the amplitude equations vanish, and, as expected, the coefficients reduce to the form obtained in [1] for internal modes.

### 5.2. Discussion of the nonlinear motion and recipe for the calculation of the amplitude equation coefficients

As in the case of internal modes, two intrinsically different cases  $\delta_T \neq 0$  and  $\delta_T = 0$  must be distinguished. According to (107) with (64), (79) and (A19),  $\delta_T$  is trilinear in the perturbational quantities  $\Phi_1$ ,  $\mathfrak{A}_1$  etc. Hence,  $\delta_T$  vanishes if the equilibria under consideration have an ignorable coordinate  $\theta$  ( $\partial p_0 / \partial \theta \equiv 0$  etc.) like the toroidal angle in tokamaks, and if the linear perturbations  $\Phi_1$ ,  $\mathfrak{A}_1$  etc. have a  $\theta$ -dependence  $\sim \sin n\theta$  or  $\sim \cos n\theta$  with  $n \neq 0$ . Thus, we have  $\delta_T = 0$  for all nonaxisymmetric modes in tokamaks (e.g. kink modes) and even for axisymmetric modes which are up/down antisymmetric. On the other hand, in stellarators we will have  $\delta_T \neq 0$  in general.

#### Typical stellarator case $\delta_T \neq 0$

For  $\delta_T \neq 0$ , already (105) is a nonlinear amplitude equation. According to (24) and (26)

$$\phi = \varepsilon A_1(T) \Phi_1 + \text{higher order terms.}$$

Therefore, if we introduce

$$A = \varepsilon A_1 \quad (114)$$

as amplitude of the marginal mode in (105) and return from  $T$  to  $t$  using (22) with (20), we obtain

$$d^2 A / dt^2 = \tau \gamma_T^2 A + (3/2) \delta_T A^2. \quad (115)$$

The same equation was obtained for internal modes, only the calculation of  $\gamma_T^2$  and  $\delta_T$  being now different. Since (115) was in detail discussed in [1], we give now only a short summary of our earlier results. For  $\lambda > \lambda_0$  or  $\tau > 0$  resp. the instability is nonlinearly enhanced to an explosive instability if the amplitude (114) of the marginal perturbation  $\Phi_1$  has initially a certain sign. For the other sign, the instability gets nonlinearly saturated such that the plasma oscillates nonlinearly about a bifurcating

equilibrium. This bifurcation is a linear one. Even in the linearly stable regime  $\tau < 0$  the plasma is nonlinearly unstable, the stability boundary being shifted towards

$$\lambda = \lambda_0 - \left( \frac{3 \delta_T}{2 \gamma_T^2} \right) A, \quad (116)$$

where  $A$  is the perturbation amplitude which leads to nonlinear instability.

The coefficients  $\gamma_T^2$  and  $\delta_T$  of the amplitude equation (115) are obtained from (106)–(108). They depend only on the marginal mode  $\Phi_1$  and the perturbational field  $\tilde{\mathbf{B}}_1^v = \nabla \times \mathfrak{A}_1$  or  $\mathfrak{A}_1$  resp. These are obtained from the linear stability theory. For the readers convenience, the corresponding equations and boundary conditions are repeated here in the notation of this paper.

In the unperturbed plasma region bounded by  $S_{00}$  we must solve

$$\mathbf{F}_{00}(\Phi_1) = 0. \quad (27)$$

In the unperturbed vacuum region bounded by  $S_{00}$  on the inside and by  $\mathfrak{C}_{00}$  on the outside we must solve

$$\nabla \times (\nabla \times \mathfrak{A}_1) = 0, \quad \nabla \cdot \mathfrak{A}_1 = 0. \quad (47)$$

The boundary conditions are

$$\mathbf{p}_{10}(\Phi_1) + \mathbf{B}_{00} \cdot \mathbf{B}_{10}(\Phi_1) = \mathbf{B}_{00}^v \cdot (\nabla \times \mathfrak{A}_1) + \Phi_1 \cdot \nabla [P_{00}] \quad (61)$$

$$\mathbf{n}_{00} \times \mathfrak{A}_1 = -\mathbf{n}_{00} \cdot \Phi_1 \mathbf{B}_{00}^v \quad (82)$$

on  $S_{00}$  and

$$\mathbf{n}_{00} \times \mathfrak{A}_1 = 0 \quad (89)$$

on  $\mathfrak{C}_{00}$ . If the equilibrium under consideration has no surface currents, the term marked by dots in (61) vanishes.

#### Typical tokamak case $\delta_T = 0$

If  $\delta_T = 0$ , in order to satisfy (105) we set

$$\tau_1 = 0 \quad (117)$$

as in the case of internal modes and obtain

$$\tau_2 (\ddot{A}_1 - \gamma_T^2 A_1) = 2 \vartheta_T A_1^3 \quad (118)$$

from (110). Again we introduce (114) as amplitude of the marginal mode. Returning with (22) and (20) from  $T$  to  $t$  we have now  $\tau = \varepsilon^2 \tau_2 + \dots$  and obtain from (118)

$$d^2 A / dt^2 = \tau \gamma_T^2 A + 2 \vartheta_T A^3. \quad (119)$$

Also this equation was discussed at length in [1]. For  $\vartheta_T > 0$ , linear instabilities are nonlinearly enhanced to an explosive instability while the nonlinear stability boundary is shifted towards

$$\lambda = \lambda_0 - 2(\vartheta_T/\gamma_T^2)A^2. \quad (120)$$

$\lambda = \lambda_0$  is a parabolic bifurcation point for equilibria branching off into the linearly stable regime  $\tau < 0$ . For  $\vartheta_T < 0$ , there is a parabolic bifurcation of equilibria into the linearly unstable regime  $\tau > 0$ , and exponential instabilities of the linear theory are nonlinearly saturated such that the plasma oscillates about the bifurcating equilibria.

The coefficients  $\gamma_T^2$  and  $\vartheta_T$  of the amplitude equation (119) are obtained from (106) and (112) if the normalization (108) is observed. For calculating  $\gamma_T^2$  we need only  $\Phi_1$  and  $\tilde{\mathbf{A}}_1$  which are obtained as in the case  $\delta_T \neq 0$ . For calculating  $\vartheta_T$ , in addition  $\Phi_{22}$  and the perturbational field  $\mathbf{B}_{00}^\vee = \nabla \times \tilde{\mathbf{A}}_{22}$  or  $\tilde{\mathbf{A}}_{22}$  resp. must be known. Again the equations and boundary conditions determining these quantities are collected here for the readers convenience.

For  $\Phi_{22}$ , from (26), (28)–(30) and (117) we get the equation

$$\mathbf{F}_{00}(\Phi_{22}) = -\mathbf{G}_{00}(\Phi_1, \Phi_1) \quad (121)$$

which must be solved in the unperturbed plasma volume bounded by  $S_{00}$ . Owing to (47), (49) and (117),  $\tilde{\mathbf{A}}_{22}$  must satisfy the equations

$$\nabla \times (\nabla \times \tilde{\mathbf{A}}_{22}) = 0, \quad \nabla \cdot \tilde{\mathbf{A}}_{22} = 0 \quad (122)$$

in the unperturbed vacuum region bounded by  $S_{00}$  inside and  $\mathbb{C}_{00}$  outside. Using the decompositions (35), (42), (49), (63), and (78), the boundary conditions (61) and (82) can be separated such that terms with different time behaviour vanish separately. Using in addition (4), (36), (41), (43), (64), (79), (A14), and (A19) we thus obtain the boundary conditions

$$\begin{aligned} \mathbf{p}_{10}(\Phi_{22}) + \mathbf{p}_1(\Phi_1, \mathbf{p}_{10}(\Phi_1)) + \mathbf{B}_{00} \cdot (\mathbf{B}_{10}(\Phi_{22}) \\ + \mathbf{B}_1(\Phi_1, \mathbf{B}_{10}(\Phi_1))) = \mathbf{B}_{00}^\vee \cdot (\nabla \times \tilde{\mathbf{A}}_{22}) \\ + (\Phi_{22} + (1/2) \Phi_1 \cdot \nabla \Phi_1) \cdot \nabla [P_{00}] - \tilde{\Phi}_{22} \end{aligned} \quad (123)$$

with

$$\begin{aligned} \tilde{\Phi}_{22} = \mathbf{B}_{10}(\Phi_1)^2 - (\nabla \times \tilde{\mathbf{A}}_1)^2 + 2 \Phi_1 \cdot \nabla [\mathbf{p}_{10}(\Phi_1) \\ + \mathbf{B}_{00} \cdot \mathbf{B}_{10}(\Phi_1) - \mathbf{B}_{00}^\vee \cdot (\nabla \times \tilde{\mathbf{A}}_1)] \\ + \Phi_1 \Phi_1 : \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} (p_{00} + \mathbf{B}_{00}^2/2 - \mathbf{B}_{00}^{\vee 2}/2) \end{aligned} \quad (124)$$

and

$$\mathbf{n}_{00} \times \tilde{\mathbf{A}}_{22} = -\mathbf{n}_{00} \cdot (\Phi_{22} + (1/2) \Phi_1 \cdot \nabla \Phi_1) \mathbf{B}_{00}^\vee - \tilde{\mathbf{E}}_{22} \quad (125)$$

with

$$\begin{aligned} \tilde{\mathbf{E}}_{22} = \mathbf{n}_{00} \times (\Phi_1 \cdot \nabla \tilde{\mathbf{A}}_1) + \tilde{\mathbf{n}}_1 \times \tilde{\mathbf{A}}_1 + \tilde{\mathbf{n}}_1 \cdot \Phi_1 \mathbf{B}_{00}^\vee \\ + \mathbf{n}_{00} \cdot \Phi_1 (\nabla \times \tilde{\mathbf{A}}_1 + \Phi_1 \cdot \nabla \mathbf{B}_{00}^\vee) \end{aligned} \quad (126)$$

on  $S_{00}$ . Thereby, for the boundary representation (14)  $\tilde{\mathbf{n}}_1$  is defined in (69) and (72). From (89), (90), and (117) we get the boundary condition

$$\mathbf{n}_{00} \times \tilde{\mathbf{A}}_{22} = 0 \quad (127)$$

on  $\mathbb{C}_{00}$ .

If the equilibrium under consideration has no surface currents, the term marked by dots in (123) vanishes.

Comment: For nonaxisymmetric modes in tokamaks we have e.g.  $\Phi_1 \sim \cos n\theta$ ,  $\theta$  being the toroidal angle. It follows from (121), that the induced mode  $\Phi_{22}$  contains contributions  $\sim \cos 2n\theta$  and contributions which do not depend on  $\theta$ . This means that a nonaxisymmetric linear instability can nonlinearly induce an axisymmetric mode. Since according to (26)  $\Phi_1$  grows  $\sim A_1(T)$  while according to (121) the induced mode under consideration grows  $\sim A_1^2(T)$ , the latter one can take over the initial instability after a while. This way, the linear appearance of an instability can be drastically changed in the nonlinear regime.

### 5.3. Simplified calculation of $\gamma_T^2$

From (106), it is possible to derive a result for  $\gamma_T^2$  which in many cases yields a significant simplification of its calculation. This result is simply

$$\begin{aligned} \gamma_T^2 = -2\varrho_{00}^{-1} (\delta^2 W_{01}^{\text{Pl}}(\Phi_1, \Phi_1) \\ + \delta^2 W_{01}^{\text{s}}(\Phi_1, \Phi_1) + \delta^2 W_{01}^{\text{v}}(\tilde{\mathbf{A}}_1, \tilde{\mathbf{A}}_1)). \end{aligned} \quad (128)$$

The meaning of this formula is the following: through the equilibrium quantities  $p_0$ ,  $\mathbf{B}_0$ , and  $\mathbf{B}_0^\vee$  the second variation  $\delta^2 W_0$  of the energy integral depends on the driving parameter  $\lambda$ . If the marginal mode  $\Phi_1$  is inserted as a test function and  $\delta^2 W_0$  is then expanded with respect to  $\tau$ ,  $\delta^2 W_{01}^{\text{Pl}}$ , and  $\delta^2 W_{01}^{\text{s}}$  are the contributions of the plasma energy and surface energy resp. to the first order coefficient of this



expansion, i.e.

$$\delta^2 W_0^{\text{Pl}}(\Phi_1, \Phi_1) = \delta^2 W_{00}^{\text{Pl}}(\Phi_1, \Phi_1) + \tau \delta^2 W_{01}^{\text{Pl}}(\Phi_1, \Phi_1) + \dots \quad (129)$$

etc.,  $\delta^2 W_0^{\text{Pl}}$  and  $\delta^2 W_0^{\text{S}}$  being given in (A24–25).  $\delta^2 W_{01}^{\text{V}}$  is defined in the following way: Let

$$\delta^2 W_0^{\text{V}}(\tilde{\mathbf{A}}, \tilde{\mathbf{A}}) = (1/2) \int_{\text{Vac}} (\nabla \times \tilde{\mathbf{A}})^2 d\tau, \quad (130)$$

where the vacuum region extends from  $S_{00}$  to  $\mathbb{C}_0(\lambda)$  and the vacuum field potential satisfies the boundary conditions

$$\begin{aligned} \mathbf{n}_{00} \times \tilde{\mathbf{A}} &= -\mathbf{n}_{00} \cdot \Phi_1 \mathbf{B}_0^{\text{V}}(\lambda) \quad \text{on } S_{00}, \\ \mathbf{n}_0 \times \tilde{\mathbf{A}} &= 0 \quad \text{on } \mathbb{C}_0(\lambda). \end{aligned} \quad (131)$$

$\delta^2 W_{01}^{\text{V}}(\tilde{\mathbf{A}}_1, \tilde{\mathbf{A}}_1)$  is the first order coefficient in the expansion

$$\delta^2 W_0^{\text{V}}(\tilde{\mathbf{A}}, \tilde{\mathbf{A}}) = \delta^2 W_{00}^{\text{V}}(\tilde{\mathbf{A}}_1, \tilde{\mathbf{A}}_1) + \tau \delta^2 W_{01}^{\text{V}}(\tilde{\mathbf{A}}_1, \tilde{\mathbf{A}}_1) + \dots \quad (132)$$

**Proof**

Similarly as (96) we obtain

$$\begin{aligned} (\Phi_1, \mathbf{F}_{01}(\Phi_1)) &= -2 \delta^2 W_{01}^{\text{Pl}}(\Phi_1, \Phi_1) \\ &\quad - \int_{S_{00}} \mathbf{n}_{00} \cdot \Phi_1 (\mathbf{p}_{11}(\Phi_1) + \mathbf{B}_{00} \cdot \mathbf{B}_{11}(\Phi_1) \\ &\quad + \mathbf{B}_{01} \cdot \mathbf{B}_{10}(\Phi_1)) dS. \end{aligned}$$

Inserting this, (64), (79) in (106) and using (A24), we obtain

$$\begin{aligned} Q_{00} \tau^2 &= -2 \delta^2 W_{01}^{\text{Pl}}(\Phi_1, \Phi_1) - 2 \delta^2 W_{01}^{\text{S}}(\Phi_1, \Phi_1) \\ &\quad - 2 \int_{S_{00}} (\mathbf{n}_{00} \cdot \Phi_1) \mathbf{B}_{01}^{\text{V}} \cdot \tilde{\mathbf{B}}_1^{\text{V}} dS \\ &\quad + \int_{\mathbb{C}_{00}} (\mathbf{n}_{00} \times \mathbf{r}_{01} \cdot \nabla \tilde{\mathbf{A}}_1 + \mathbf{n}_{01} \times \tilde{\mathbf{A}}_1) \cdot \tilde{\mathbf{B}}_1^{\text{V}} dS. \end{aligned} \quad (133)$$

Integrating by parts and using the boundary conditions (131),  $\delta^2 W_0^{\text{V}}(\tilde{\mathbf{A}}, \tilde{\mathbf{A}})$  can now be expressed by a surface integral

$$2 \delta^2 W_0^{\text{V}}(\tilde{\mathbf{A}}, \tilde{\mathbf{A}}) = - \int_{S_{00}} (\mathbf{n}_{00} \times \tilde{\mathbf{A}}) \cdot \nabla \times \tilde{\mathbf{A}} dS. \quad (134)$$

If  $\tilde{\mathbf{A}}$  is expanded according to

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}_1 + \tau \tilde{\mathbf{a}}_1 \quad (135)$$

with (44), (83), (86), and

$$\tilde{\mathbf{A}}(\mathbf{r}_0) = \tilde{\mathbf{A}}(\mathbf{r}_{00}) + \tau \mathbf{r}_{01} \cdot \nabla \tilde{\mathbf{A}}_1|_{\mathbf{r}_{00}} + \dots$$

from (131) we obtain the boundary conditions

$$\mathbf{n}_{00} \times \tilde{\mathbf{A}}_1 = -\mathbf{n}_{00} \cdot \Phi_1 \mathbf{B}_{00}^{\text{V}}, \quad (136)$$

$$\mathbf{n}_{00} \times \tilde{\mathbf{a}}_1 = -\mathbf{n}_{00} \cdot \Phi_1 \mathbf{B}_{01}^{\text{V}} \quad \text{on } S_{00}$$

and

$$\mathbf{n}_{00} \times \tilde{\mathbf{A}}_1 = 0, \quad (137)$$

$$\mathbf{n}_{00} \times \tilde{\mathbf{a}}_1 = -(\mathbf{r}_{01} \cdot \nabla \tilde{\mathbf{A}}_1 + \mathbf{n}_{01} \times \tilde{\mathbf{A}}_1) \quad \text{on } \mathbb{C}_{00}.$$

Obviously  $\tilde{\mathbf{A}}_1$  is the marginal vector potential defined by (47), (82), and (89) while, in general,  $\tilde{\mathbf{a}}_1 \neq \tilde{\mathbf{A}}_1$  due to different boundary conditions.

Inserting (135) in (134), by comparison with (132) we obtain

$$\begin{aligned} 2 \delta^2 W_{01}^{\text{V}} &= - \int_{S_{00}} [(\mathbf{n}_{00} \times \tilde{\mathbf{a}}_1) \cdot \nabla \times \tilde{\mathbf{A}}_1 \\ &\quad + (\mathbf{n}_{00} \times \tilde{\mathbf{A}}_1) \cdot \nabla \times \tilde{\mathbf{a}}_1] dS. \end{aligned} \quad (138)$$

Another integration by parts yields

$$\begin{aligned} &\int_{V_{\text{ac}}^{00}} (\nabla \times \tilde{\mathbf{A}}_1) \cdot (\nabla \times \tilde{\mathbf{a}}_1) d\tau \\ &= - \int_{S_{00}} (\mathbf{n}_{00} \cdot \tilde{\mathbf{A}}_1) \cdot (\nabla \times \tilde{\mathbf{a}}_1) dS + \int_{\mathbb{C}_{00}} (\mathbf{n}_{00} \times \tilde{\mathbf{A}}_1) \cdot (\nabla \times \tilde{\mathbf{a}}_1) dS \\ &= - \int_{S_{00}} (\mathbf{n}_{00} \times \tilde{\mathbf{a}}_1) \cdot (\nabla \times \tilde{\mathbf{A}}_1) dS + \int_{\mathbb{C}_{00}} (\mathbf{n}_{00} \times \tilde{\mathbf{a}}_1) \cdot (\nabla \times \tilde{\mathbf{A}}_1) dS \end{aligned}$$

or

$$\begin{aligned} - \int_{S_{00}} (\mathbf{n}_{00} \times \tilde{\mathbf{A}}_1) \cdot (\nabla \times \tilde{\mathbf{a}}_1) dS &= - \int_{S_{00}} (\mathbf{n}_{00} \times \tilde{\mathbf{a}}_1) \cdot (\nabla \times \tilde{\mathbf{A}}_1) dS \\ &\quad + \int_{\mathbb{C}_{00}} (\mathbf{n}_{00} \times \tilde{\mathbf{a}}_1) \cdot (\nabla \times \tilde{\mathbf{A}}_1) dS \end{aligned} \quad (139)$$

since  $\mathbf{n}_{00} \times \tilde{\mathbf{A}}_1 = 0$  on  $\mathbb{C}_{00}$ . Inserting (139) in (138), with (136)–(137) and  $\tilde{\mathbf{B}}_1^{\text{V}} = \nabla \times \tilde{\mathbf{A}}_1$  we finally obtain

$$\begin{aligned} 2 \delta^2 W_{01}^{\text{V}} &= 2 \int_{S_{00}} (\mathbf{n}_{00} \cdot \Phi_1) \mathbf{B}_{01}^{\text{V}} \cdot \tilde{\mathbf{B}}_1^{\text{V}} dS \\ &\quad - \int_{\mathbb{C}_{00}} (\mathbf{r}_{01} \cdot \nabla \tilde{\mathbf{A}}_1 + \mathbf{n}_{01} \times \tilde{\mathbf{A}}_1) \cdot \tilde{\mathbf{B}}_1^{\text{V}} dS \end{aligned}$$

which with (133) confirms our initial statement.  $\square$

#### 5.4. Comment

Integrating (105) and (118) with respect to  $T$  yields equations which have the structure of an energy equation. This suggests that time-integrated versions of the amplitude equations can directly be obtained from the equation of energy conservation

$$K + W = \text{const}, \quad K = \int_{\text{Pl}} \varrho v^2/2 d\tau, \quad (140)$$

$$W = \int_{\text{Pl}} (3p/2 + B^2/2) d\tau + \int_{\text{Vac}} B^2/2 d\tau$$

by inserting the expansions (21) and (24) with (23) etc.,  $\Phi_1, \Phi_2, \dots$  being the solutions of the boundary value problems treated in the previous sections. This is in fact the case, since e.g. for  $\tau_1 \neq 0$  according to (19), (20), (23), (24), and (108)

$$K = (1/2) \varrho_{00} \dot{A}_1^2 \tau_1 \varepsilon^3 + o(\varepsilon^3).$$

Therefore, to third order in  $\varepsilon$  (140) delivers an expression for  $\dot{A}_1^2$  which is just what one obtains from (105). However, this access to the desired amplitude equations is not as direct and fast as it would appear at first glance. For  $\tau_1 \neq 0$  e.g.,  $\varepsilon^3$  is the lowest order to which dynamical effects appear in (140), and hence, for obtaining the equivalent of (105), one must already employ the expansions (21) etc. up to third order, while for deriving the equivalent of (118) one would even have to calculate  $p_4$ ,  $B_4$  and  $B_4^y$ . As we know, these higher order terms do not appear in the final results, and thus, tedious integrations by parts must be carried out in order to show that the contributions of these higher order terms cancel.

Nevertheless one would at least hope for a direct approach to (128) since expanding  $W$  with respect to  $\xi$  exhibits directly the corresponding energy variation:

$$W = W^* + \delta^1 W + \delta^2 W + \delta^3 W + \dots$$

Again this expectation is frustrated due to difficulties arising from nonlinearity. Let us consider internal modes for simplicity. For these, we have generally

$$\begin{aligned} \delta^1 W &= \delta^1 W^{\text{Pl}} = \int_{\text{Pl}} \xi \cdot (\nabla p - j \times B) dS \\ &\quad - \int_S n \cdot \xi (p + B^2/2) dS \end{aligned}$$

and

$$\delta^1 W_0 = - \int_{S_0} n_0 \cdot \xi (p_0 + B_0^2/2) dS$$

for variations about an equilibrium state as considered. Only in linear stability theory we have  $n_0 \cdot \xi = 0$  and  $\delta^1 W_0 = 0$ . The nonlinear boundary condition for internal modes is  $n_{00} \cdot v = 0$  or ([1])

$$n_{00} \cdot \Phi_i = 0, \quad i = 1, 2, 3, \dots$$

Thus, according to (38), we have

$$n_{00} \cdot \xi_2 = (1/2) n_{00} \cdot (\Phi_1 \cdot \nabla \Phi_1)$$

which cannot assumed to be zero, and similarly  $n_{00} \cdot \xi_3 \neq 0$  etc. Thus, expanding (140) we have to

deal with terms like  $\delta^1 W_{00}(\xi_2)$ ,  $\delta^1 W_{00}(\xi_3)$ ,  $\delta^1 W_{01}(\xi_2)$  etc. which conceal the simple relation (128) and necessitate again tedious calculations for its derivation.

## 6. Higher Order Corrections

In reality, the time evolution of an instability can follow the theoretical course of exponential or explosive growth only for restricted time since otherwise energy conservation would be strongly violated. This problem is settled by the transition of an exponential instability of the linear theory into a nonlinear oscillation. If, to lowest nonlinear order, an explosive instability is found, energy conservation cannot be brought about by the induced modes  $\tau_1 A_1(T) \Phi_{21}(\mathbf{r})$ ,  $(1/2) A_1^2(T) \Phi_{22}(\mathbf{r})$  etc. since these are directly coupled to the primary mode  $A_1(T) \Phi_1(\mathbf{r})$  and become explosive at the same time. Thus, only nonlinear effects of higher order can help in this case which must necessarily become important after some time.

We can study the effect of higher order corrections quantitatively for  $\delta_T \neq 0$  since in this case (110) is a higher order correction for (105). Eq. (110) was already qualitatively discussed in [1]. If the solution  $A_1(T)$  of (105) is a nonlinear oscillation, eq. (110) has the structure of a Hills equation with periodic forcing term and may exhibit parametric instability. On the other hand, if (105) predicts explosive instability,  $A_2(T)$  as calculated from (110) may counteract the explosive decay and stabilize the motion. There is of course also the possibility that  $A_1(T)$  and  $A_2(T)$  aim in the same direction. All qualitative conclusions of this kind can quantitatively be confirmed by replacing the system of (105) and (110) by a single amplitude equation to be derived immediately which is equivalent to this system up to terms of order  $\varepsilon^3$ .

Firstly, we define a higher order amplitude of the marginal mode  $\Phi_1$  by inspecting the equation

$$\begin{aligned} \Phi &= (\varepsilon A_1 + \varepsilon^2 A_2) \Phi_1 \\ &\quad + \varepsilon^2 (A_1 \Phi_{21} + (A_1^2/2) \Phi_{22}) + \dots \end{aligned} \quad (141)$$

which follows from (24), (26), (29), and (91a). Obviously, the proper definition is

$$A = \varepsilon A_1 + \varepsilon^2 A_2. \quad (142)$$

In order to obtain an equation for this amplitude, we multiply (105) by  $\varepsilon^2$  and add it to (110) multi-

plied by  $\varepsilon^3$ . Simultaneously, from the third bracket on the left hand side of (110) we eliminate  $\ddot{A}_1$  and  $\dot{A}_1$  using (105) and its time integral

$$\dot{A}_1^2 = \gamma_T^2 A_1^2 + (\delta_T/\tau_1) A_1^3. \quad (143)$$

Note that in (143) an integration constant has been chosen such that  $\dot{A}_1 = 0$  for  $A_1 = 0$  as is the case for exponential instabilities when  $t \rightarrow -\infty$ . Carrying out the steps just described we arrive at the equation

$$\begin{aligned} & \tau_1 \varepsilon^2 (\ddot{A}_1 - \gamma_T^2 A_1) + \tau_2 \varepsilon^3 (\ddot{A}_1 - \gamma_T^2 A_1) + \tau_1 \varepsilon^3 (\ddot{A}_2 - \gamma_T^2 A_2) \\ & + \tau_1 \varepsilon^3 [\gamma_T^2 (\tau_1 a_1 A_1 + a_2 A_1^2 + b A_1^3) + (3/2) \delta_T (a_1 A_1^2 + a_2 A_1^3/\tau_1 + 2 b A_1^3/(3 \tau_1))] \\ & = (3/2) \delta_T \varepsilon^2 A_1^2 + 3 \delta_T \varepsilon^3 A_1 A_2 + \tau_1 \varepsilon^3 (\tau_1 c_1 A_1 + c_2 A_1^2) + 2 \vartheta_T \varepsilon^3 A_1^3. \end{aligned}$$

According to the derivation, (143) is valid up to terms of order  $\varepsilon^3$  inclusively. Now, from (20) and (142) we get

$$\begin{aligned} \tau (\ddot{A} - \gamma_T^2 A) &= \tau_1 \varepsilon^2 (\ddot{A}_1 - \gamma_T^2 A_1) + \tau_1 \varepsilon^3 (\ddot{A}_2 - \gamma_T^2 A_2) + \tau_2 \varepsilon^3 (\ddot{A}_1 - \gamma_T^2 A_1) + O(\varepsilon^4), \\ a_1 \gamma_T^2 \tau^2 A &+ (a_2 \gamma_T^2 + (3/2) a_1 \delta_T + b \gamma_T^2) \tau A^2 + (3 a_2/2 + b) \delta_T A^3 \\ &= \varepsilon^3 \tau_1 [\gamma_T^2 (\tau_1 a_1 A_1 + a_2 A_1^2 + b A_1^3) + (3/2) \delta_T (a_1 A_1^2 + a_2 A_1^3/\tau_1 + 2 b A_1^3/(3 \tau_1))] + O(\varepsilon^4), \\ (3/2) \delta_T A^2 &= (3/2) \delta_T \varepsilon^2 A_1^2 + 3 \delta_T \varepsilon^3 A_1 A_2 + O(\varepsilon^4), \\ \tau (\tau c_1 A + c_2 A^2) &= \tau_1 \varepsilon^3 (\tau_1 c_1 A_1 + c_2 A_1^2) + O(\varepsilon^4), \\ 2 \vartheta_T A^3 &= 2 \vartheta_T \varepsilon^3 A_1^3 + O(\varepsilon^4). \end{aligned}$$

Inserting these relations in (143) and returning from  $T$  to  $t$  with (22), we finally obtain

$$\begin{aligned} d^2 A/dt^2 &= \tau \gamma_T^2 [1 - (a_1 - c_1/\gamma_T^2) \tau] A + (3/2) \delta_T \{1 - [a_1 + 2(a_2 + b) \gamma_T^2/(3 \delta_T) \\ &- 2 c_2/(3 \delta_T)] \tau\} A^2 + [2 \vartheta_T - 3(a_2 + b) \delta_T/2] A^3. \end{aligned} \quad (144)$$

As was already mentioned, (144) is equivalent to the system (105) and (110), errors due to the omission of higher order terms being of the same order. For the case  $\delta_T = 0$ , to the lowest nonlinear order we obtained the amplitude equation (115), and hence (144) is a higher order correction of this equation. It contains the following correction terms: the linear growth rate  $\tau \gamma_T^2$  in (115) is corrected by terms of order  $\tau^2$ ; the nonlinear coefficient  $\delta_T$  in (115) is again corrected by terms of order  $\tau$ . Finally, (144) contains an additional correction term which is of higher order in the amplitude.

(144) may be integrated once with respect to  $t$  yielding

$$\begin{aligned} (dA/dt)^2 + V(A) &= \text{const}, \\ V(A) &= -(\tau \gamma_T^{*2} A^2 + \delta_T^* A^3 + \vartheta_T^* A^4), \end{aligned} \quad (145)$$

where  $\gamma_T^*$ ,  $\delta_T^*$  and  $\vartheta_T^*$  are easily obtained from (144). Like in [1], we may choose the sign of  $\Phi_1$  such that

without loss of generality we have

$$\delta_T^* \geq 0.$$

Depending on the sign of  $\vartheta_T^*$ ,  $V(A)$  may then either have the shape of Figure 1 or that of Figure 2. If it is that of Figure 1, then the explosive instability occurring for  $A > 0$  according to the lower order equation (115) is getting saturated and becomes a nonlinear oscillation about a new equilibrium position

which appears at  $A = A^{(2)}$ . The bifurcation diagram which corresponds to this situation is shown in Figure 3. If  $V(A)$  has the shape of Figure 2, then the explosive instability obtained from (115) for  $A > 0$  is even enhanced, while the bifurcating equilibrium position at  $A = -A^{(1)}$  becomes nonlinearly unstable. The corresponding bifurcation diagram is

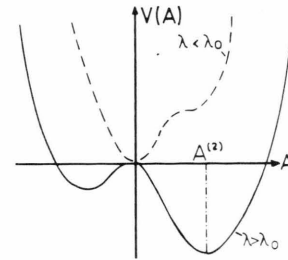


Fig. 1. Potential  $V(A)$ , Eq. (145), for  $\vartheta_T^* < 0$  in a (linearly) stable ( $\lambda < \lambda_0$ ) and in an unstable ( $\lambda > \lambda_0$ ) situation.

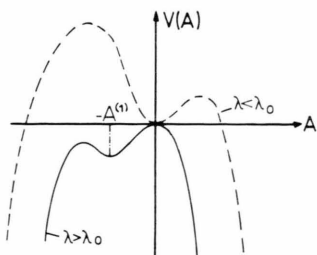


Fig. 2. Potential  $V(A)$ , Eq. (145), for  $\vartheta_T^* > 0$  in a (linearly) stable ( $\lambda < \lambda_0$ ) and in an unstable ( $\lambda > \lambda_0$ ) situation.

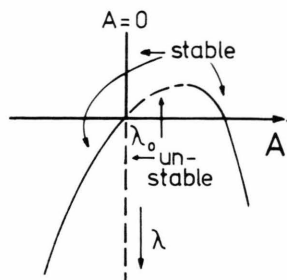


Fig. 3. Bifurcation diagram corresponding to Figure 1.

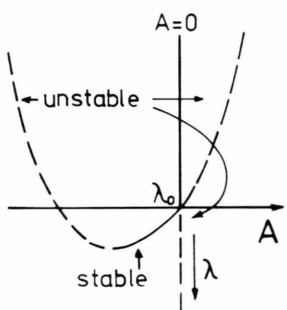


Fig. 4. Bifurcation diagram corresponding to Figure 2.

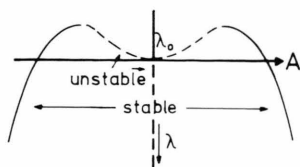


Fig. 5. Bifurcation diagram for higher order equation corresponding to (118) with  $\vartheta_T > 0$ .

shown in Figure 4. In the last case it would become necessary to go even to higher orders if one wants to obtain an amplitude equation which is not in contradiction with energy conservation.

In the case  $\delta_T = 0$ , (144) leads essentially back to (119) as one would expect: For small  $A$  when the

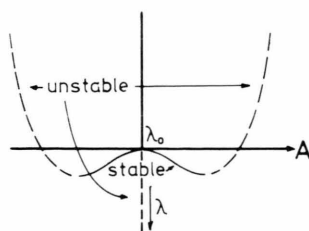


Fig. 6. Bifurcation diagram for higher order equation corresponding to (118) with  $\vartheta_T < 0$ .

nonlinear terms can be neglected, (144) differs from (115) only by a small correction of the growth rate which can also be obtained from linear theory by going to higher orders in  $\tau$ . If  $A$  has the order of magnitude of bifurcating equilibrium values, according to (144) we have  $A \sim \tau^{1/2}$ . Then, up to terms of order  $\tau^{3/2}$  (144) coincides with (119). Terms correcting the linear and nonlinear contributions in (144) are  $\sim \tau^{5/2}$  and are small as compared with the  $\tau^{3/2}$  terms contained in (119).

It is certainly possible however tedious to derive also a higher order amplitude equation for the case  $\delta_T = 0$  which corrects (119). If  $\delta_T = 0$  due to the symmetry of the problem, then also the coefficients of the next order will vanish and one will have to go two orders further for obtaining a correction of (119). It is obvious that in addition to slight corrections of the coefficients  $\gamma_T^2$  and  $\vartheta_T$  this equation will contain an  $A^5$  term on the right hand side. Again, nonlinear stabilization or destabilization will be possible. If a previously unstable plasma is getting stabilized, we have the bifurcation diagram of Fig. 5 while that of Fig. 6 is obtained, if the higher order effects drive stable bifurcating equilibria nonlinearly unstable.

## 7. Sketch of an Alternative Approach and Problems of Existence for Bifurcating Equilibria

### 7.1. Alternative approach

In order to integrate (17), in (23) we have introduced the Eulerian variable  $\varphi$  the expansion functions of which obey the differential equations (27), (28), and (31). Replacing  $\varphi$  by the plasma shift  $\xi$  we come to a Lagrangian description. This can be achieved either by inserting (38) in the latter equations or quite directly in the following way.

Using (13), the last three of (17) can be integrated<sup>1</sup> to yield

$$\begin{aligned}\hat{q}(\mathbf{r}, t) &:= q(\mathbf{r} + \boldsymbol{\xi}(\mathbf{r}, t), t) = q(\mathbf{r}, 0)/D, \\ \hat{p}(\mathbf{r}, t) &= p(\mathbf{r}, 0)/D^{5/3}, \\ \hat{\mathbf{B}}_k(\mathbf{r}, t) &= D_{km} B_m(\mathbf{r}, 0)/D,\end{aligned}\quad (146)$$

where Einsteins summation convention is used and where

$$D_{ik} = \delta_{ik} + \partial \xi_i(\mathbf{r}, t)/\partial x_k, \quad D = \det D_{ik}. \quad (147)$$

Equations (147) are usually obtained from integral conservation laws for mass, entropy and magnetic field flux [8]–[9]. A direct rederivation from the corresponding differential conservation laws is given in [10]. Having the full nonlinear solutions (146) at hand,  $p$ ,  $q$  and  $\mathbf{B}$  can be eliminated from (17). For this purpose, we define

$$\mathbf{r}' = \mathbf{r} + \boldsymbol{\xi}(\mathbf{r}, t) \quad (148)$$

and, using  $\mathbf{j} \times \mathbf{B} = -\nabla B^2/2 + \mathbf{B} \cdot \nabla \mathbf{B}$ , the first of (17) can be written

$$\hat{q} \, d\mathbf{r}'(\mathbf{r}', t)/dt = -\nabla' \cdot (\hat{p} + \hat{\mathbf{B}}^2/2) + \hat{\mathbf{B}} \cdot \nabla' \hat{\mathbf{B}}. \quad (149)$$

From (13) and (147)–(148) we get

$$d\mathbf{r}'(\mathbf{r}', t)/dt = \partial^2 \boldsymbol{\xi}(\mathbf{r}, t)/\partial t^2. \quad (150)$$

Furthermore, according to (147) and (148)

$$\partial \hat{p}/\partial x_l = (\partial \hat{p}/\partial x'_k) \partial x'_k/\partial x_l = D_{kl} \partial \hat{p}/\partial x'_k. \quad (151)$$

Denoting the inverse matrix of  $D_{kl}$  by  $D_{kl}^{-1}$ , from (151) we obtain

$$\partial \hat{p}/\partial x'_l = D_{li}^{-1} \partial \hat{p}/\partial x_l \quad (152)$$

and similarly

$$\partial \hat{\mathbf{B}}_i/\partial x'_k = D_{ik}^{-1} \partial \hat{\mathbf{B}}_i/\partial x_l. \quad (153)$$

Inserting (150) and (152) in (149), again using summation convention we obtain

$$\begin{aligned}\hat{q} \, \partial^2 \xi_i/\partial t^2 &= -D_{li}^{-1} \partial (\hat{p} + \hat{\mathbf{B}}^2/2)/\partial x_l \\ &\quad + D_{lk}^{-1} \hat{\mathbf{B}}_k \partial \hat{\mathbf{B}}_i/\partial x_l.\end{aligned}\quad (154)$$

If the solutions (146) are inserted in (153), finally the desired nonlinear equation for  $\boldsymbol{\xi}$  is obtained:

$$\begin{aligned}q \, \partial^2 \xi_i/\partial t^2 &= -D \, D_{li}^{-1} \partial (D^{-5/3} p + D^{-2} D_{kr} D_{ks} B_r B_s)/ \\ &\quad \cdot \partial x_l + B_l \partial (D^{-1} D_{ik} B_k)/\partial x_l,\end{aligned}\quad (155)$$

where for  $p(\mathbf{r}, 0)$  we have simply written  $p$  etc.

With (22) and (25) from (155) one obtains equations for  $\xi_1, \xi_2$  etc. which are equivalent to (27), (28) and (31). Although at first glance this Lagrangian approach might appear much simpler than the Eulerian one adopted in this paper, it turns out that a comparable amount of calculations is necessary until a practical form of the reduced equations is achieved.

Concerning the boundary conditions we have adopted a Lagrangian description already from the beginning. Only at the very end we have switched from  $\boldsymbol{\xi}$  to  $\boldsymbol{\varphi}$  in order to get a connection with our plasma description. Thus, for a full Lagrangian description we can use the reduced boundary conditions as derived and have only to keep with  $\boldsymbol{\xi}$ .

## 7.2. Existence problems

Putting  $\partial^2 \xi_i/\partial t^2 \equiv 0$ , (115) reduces to an equilibrium equation. With  $p = p_0(\mathbf{r}, \lambda)$ ,  $\mathbf{B} = \mathbf{B}_0(\mathbf{r}, \lambda)$  and  $\boldsymbol{\xi} \rightarrow \boldsymbol{\xi}_1$  for  $\lambda \rightarrow \lambda_0$  it is the exact plasma equation for the bifurcating equilibria found in Sections 4–5. It must be supplemented by the vacuum equations (18), by the boundary conditions (8) and (15) on  $S_0$  (with  $\partial \boldsymbol{\varphi}/\partial t \equiv 0$  and  $\mathbf{v} \equiv 0$ , (7) is satisfied automatically) and by (16) on  $\mathcal{C}_0$ .

Even in axisymmetric equilibria e.g. tokamaks, for nonaxisymmetric modes the bifurcating equilibria have the geometrical structure of stellarators. For exact MHD equilibria of this kind the existence is rather doubtful [11]. Problems arise due to the fact that only the  $\mathbf{j}$ -component perpendicular to  $\mathbf{B}$  enters the equilibrium equations and that  $\mathbf{j}_\perp = \mathbf{j} \cdot \mathbf{B}/B$  blows up. No physical mechanism is seen by which the singularities of  $\mathbf{j}_\perp$  could be avoided.

For bifurcating equilibria the situation appears somewhat better. According to (146),  $\mathbf{j}_\perp$  is given by

$$\mathbf{j}_\perp = (B_i/B) \, \varepsilon_{ijk} \partial (D^{-1} D_{km} B_m)/\partial x_j, \quad (156)$$

$\varepsilon_{ijk}$  being the Levi-Civita Tensor. Since the unperturbed  $\mathbf{B}$ -field is everywhere regular, singularities of  $\mathbf{j}_\perp$  can only come about through singularities of  $D_{km}$ ,  $\partial D/\partial x_j$  or  $\partial D_{km}/\partial x_j$  (the case  $D = 0$  can be excluded since  $D \rightarrow 1$  for  $\boldsymbol{\xi} \rightarrow 0$ ). We can imagine that the bifurcating equilibria originate from an existing unstable axisymmetric equilibrium via regular initial perturbations leading to instability. If the system contains some friction, it may ultimately settle down in a neighbouring equilibrium. Since the initial perturbation is regular, here in contrast a

<sup>1</sup> P. Merkel and A. Schlüter have led the authors attention to [8]–[9] and the solutions (146), which is gratefully acknowledged.



physical mechanism would be required to make the abovementioned quantities singular. However, this consideration is only at plausible and cannot replace a strict proof. In the end, the question is left open whether the bifurcating equilibria obtained from reductive perturbation theory are an indication for the existence of exact bifurcating equilibria or whether they come about only by “tyranny”<sup>1</sup> of the perturbation method while exact neighbouring equilibria do not exist.

### Summary

The amplitude equations obtained for linearly marginal external modes are qualitatively the same as the ones obtained for internal modes. Thus, depending on the symmetry of the problem, again two cases are possible. In the typical stellarator case, generally an explosive instability appears in the linearly unstable regime. In the typical tokamak case, either an explosive instability or a nonlinear oscillation about a stable bifurcating equilibrium is possible. If explosive instabilities occur, the stability limit is nonlinearly shifted into the linearly stable regime. Nonlinearly, induced modes may drastically change the appearance of the initial instability.

Like for internal modes, there is a close relation between the type of nonlinear motion and the bifurcation of equilibria from the marginally stable state, this bifurcation being always observed. It cannot be concluded from the perturbation expansion whether these bifurcating equilibria exist to all orders or whether they are quasi-equilibria corresponding to slowly varying plasma states. When an explosive instability is predicted by the nonlinear theory, the law of energy conservation will be violated after some time. This problem can be settled by taking into account higher order corrections. The way how it is settled in the framework of reductive perturbation theory is always such that after some time of rapid instability growth the plasma oscillates ultimately about a bifurcating equilibrium.

### Appendix I: List of Definitions

$\mathbf{j}$ ,  $\mathbf{B}$ ,  $\boldsymbol{\varphi}$ ,  $\boldsymbol{\chi}$ ,  $\boldsymbol{\Phi}$ ,  $\boldsymbol{\xi}$  etc. and  $p$ ,  $\chi$  resp. are vector and scalar fields resp.  $\mathbf{B}_1$ ,  $\mathbf{B}_{10}$ ,  $\mathbf{F}_0$ ,  $\mathbf{F}_{00}$ ,  $\mathbf{G}_{00}$  etc. are vector operators  $\mathbf{p}_1$ ,  $\mathbf{p}_{10}$  etc. are scalar operators.  $[P_{0i}]$  are

<sup>1</sup> This formulation was communicated to the author by G. Nicolis.

scalar functions defined on the undisplaced plasma-surface  $S_{00}$ .

$$\dot{f} = df/dT, \quad (\text{A1})$$

$$(\boldsymbol{\varphi}, \boldsymbol{\chi}) = \int_{\text{Pl}_{00}} \boldsymbol{\varphi} \cdot \boldsymbol{\chi} \, d\tau, \quad (\text{A2})$$

$$\mathbf{B}_1(\boldsymbol{\varphi}, \boldsymbol{\chi}) = \nabla \times (\boldsymbol{\varphi} \times \boldsymbol{\chi}), \quad (\text{A3})$$

$$\mathbf{p}_1(\boldsymbol{\varphi}, \boldsymbol{\chi}) = -(\boldsymbol{\varphi} \cdot \nabla \boldsymbol{\chi} + (5/3) \boldsymbol{\chi} \cdot \nabla \boldsymbol{\varphi}), \quad (\text{A4})$$

$$\mathbf{B}_{1i}(\boldsymbol{\varphi}) = \mathbf{B}_1(\boldsymbol{\varphi}, \mathbf{B}_{0i}), \quad i = 0, 1, 2, \quad (\text{A5})$$

$$\mathbf{p}_{1i}(\boldsymbol{\varphi}) = \mathbf{p}_1(\boldsymbol{\varphi}, p_{0i}), \quad i = 0, 1, 2, \quad (\text{A6})$$

$$\mathbf{F}_0(\boldsymbol{\xi}) = \mathbf{j}_0 \times \mathbf{B}_1(\boldsymbol{\xi}, \mathbf{B}_0) + [\nabla \times \mathbf{B}_1(\boldsymbol{\xi}, \mathbf{B}_0)] \times \mathbf{B}_0 - \nabla \mathbf{p}_1(\boldsymbol{\xi}, p_0), \quad (\text{A7})$$

$$\mathbf{F}_{00}(\boldsymbol{\varphi}) = \mathbf{j}_{00} \times \mathbf{B}_{10}(\boldsymbol{\varphi}) + [\nabla \times \mathbf{B}_{10}(\boldsymbol{\varphi})] \times \mathbf{B}_{00} - \nabla \mathbf{p}_{10}(\boldsymbol{\varphi}), \quad (\text{A8})$$

$$\mathbf{F}_{01}(\boldsymbol{\varphi}) = \mathbf{j}_{00} \times \mathbf{B}_{11}(\boldsymbol{\varphi}) + [\nabla \times \mathbf{B}_{11}(\boldsymbol{\varphi})] \times \mathbf{B}_{00} - \nabla \mathbf{p}_{11}(\boldsymbol{\varphi}) + \mathbf{j}_{01} \times \mathbf{B}_{10}(\boldsymbol{\varphi}) + [\nabla \times \mathbf{B}_{10}(\boldsymbol{\varphi})] \times \mathbf{B}_{01}, \quad (\text{A9})$$

$$\mathbf{F}_{02}(\boldsymbol{\varphi}) = \mathbf{j}_{00} \times \mathbf{B}_{12}(\boldsymbol{\varphi}) + [\nabla \times \mathbf{B}_{12}(\boldsymbol{\varphi})] \times \mathbf{B}_{00} - \nabla \mathbf{p}_{12}(\boldsymbol{\varphi}) + \mathbf{j}_{02} \times \mathbf{B}_{10}(\boldsymbol{\varphi}) + [\nabla \times \mathbf{B}_{10}(\boldsymbol{\varphi})] \times \mathbf{B}_{02} + \mathbf{j}_{01} \times \mathbf{B}_{11}(\boldsymbol{\varphi}) + [\nabla \times \mathbf{B}_{11}(\boldsymbol{\varphi})] \times \mathbf{B}_{01}, \quad (\text{A10})$$

$$\mathbf{G}_{00}(\boldsymbol{\varphi}, \boldsymbol{\chi}) = \mathbf{j}_{00} \times \mathbf{B}_1(\boldsymbol{\varphi}, \mathbf{B}_{10}(\boldsymbol{\chi})) + [\nabla \times \mathbf{B}_1(\boldsymbol{\varphi}, \mathbf{B}_{10}(\boldsymbol{\chi}))] \times \mathbf{B}_{00} - \nabla \mathbf{p}_1(\boldsymbol{\varphi}, \mathbf{p}_{10}(\boldsymbol{\chi})) + [\nabla \times \mathbf{B}_{10}(\boldsymbol{\varphi})] \times \mathbf{B}_{10}(\boldsymbol{\chi}) + [\nabla \times \mathbf{B}_{10}(\boldsymbol{\chi})] \times \mathbf{B}_{10}(\boldsymbol{\varphi}), \quad (\text{A11})$$

$$\mathbf{G}_{01}(\boldsymbol{\varphi}, \boldsymbol{\varphi}) = \mathbf{j}_{00} \times \mathbf{B}_1(\boldsymbol{\varphi}, \mathbf{B}_{11}(\boldsymbol{\varphi})) + [\nabla \times \mathbf{B}_1(\boldsymbol{\varphi}, \mathbf{B}_{11}(\boldsymbol{\varphi}))] \times \mathbf{B}_{00} - \nabla \mathbf{p}_1(\boldsymbol{\varphi}, \mathbf{p}_{11}(\boldsymbol{\varphi})) + \mathbf{j}_{01} \times \mathbf{B}_1(\boldsymbol{\varphi}, \mathbf{B}_{10}(\boldsymbol{\varphi})) + [\nabla \times \mathbf{B}_1(\boldsymbol{\varphi}, \mathbf{B}_{10}(\boldsymbol{\varphi}))] \times \mathbf{B}_{01} + 2[\nabla \times \mathbf{B}_{10}(\boldsymbol{\varphi})] \times \mathbf{B}_{11}(\boldsymbol{\varphi}) + 2[\nabla \times \mathbf{B}_{11}(\boldsymbol{\varphi})] \times \mathbf{B}_{10}(\boldsymbol{\varphi}), \quad (\text{A12})$$

$$\mathbf{H}_{00}(\boldsymbol{\varphi}, \boldsymbol{\varphi}, \boldsymbol{\varphi}) = \mathbf{j}_{00} \times \mathbf{B}_1(\boldsymbol{\varphi}, \mathbf{B}_1(\boldsymbol{\varphi}, \mathbf{B}_{10}(\boldsymbol{\varphi}))) + [\nabla \times \mathbf{B}_1(\boldsymbol{\varphi}, \mathbf{B}_1(\boldsymbol{\varphi}, \mathbf{B}_{10}(\boldsymbol{\varphi})))] \times \mathbf{B}_{00} - \nabla \mathbf{p}_1(\boldsymbol{\varphi}, \mathbf{p}_1(\boldsymbol{\varphi}, \mathbf{p}_{10}(\boldsymbol{\varphi}))) + 3[\nabla \times \mathbf{B}_{10}(\boldsymbol{\varphi})] \times \mathbf{B}_1(\boldsymbol{\varphi}, \mathbf{B}_{10}(\boldsymbol{\varphi})) + 3[\nabla \times \mathbf{B}_1(\boldsymbol{\varphi}, \mathbf{B}_{10}(\boldsymbol{\varphi}))] \times \mathbf{B}_{10}(\boldsymbol{\varphi}), \quad (\text{A13})$$

$$[P_{00}] = B_{00}^2/2 - (B_{00}^2/2 + p_{00}), \quad (\text{A14})$$

$$[P_{01}] = B_{00}^2 \cdot B_{01}^2 - (B_{00} \cdot B_{01} + p_{01}), \quad (\text{A15})$$

$$[P_{02}] = B_{00}^2 \cdot B_{02}^2 - (B_{00} \cdot B_{02} + p_{02}) + B_{01}^2/2 - B_{01}^2/2, \quad (\text{A16})$$

$$[P_{03}] = B_{00}^2 \cdot B_{03}^2 - (B_{00} \cdot B_{03} + p_{03}) + B_{01}^2 \cdot B_{02}^2 - B_{01} \cdot B_{02}, \quad (\text{A17})$$

$$[P_1] = B_{00}^2 \cdot B_1^2 - (B_{00} \cdot B_1 + p_1), \quad (\text{A18})$$

$$[\tilde{P}_1] = B_{00}^2 \cdot \tilde{B}_1^2 - (B_{00} \cdot \tilde{B}_1 + \tilde{p}_1), \quad (\text{A19})$$

$$[P_2] = B_{00}^2 \cdot B_2^2 - (B_{00} \cdot B_2 + p_2) + B_1^2/2 - B_1^2/2, \quad (\text{A20})$$

$$[\tilde{P}_{2i}] = B_{00}^2 \cdot \tilde{B}_{2i}^2 - (B_{00} \cdot \tilde{B}_{2i} + \tilde{p}_{2i}), \quad i = 0, 1, \quad (\text{A21})$$

$$[\tilde{P}_{22}] = \mathbf{B}_{00}^y \cdot \tilde{\mathbf{B}}_{22}^y - (\mathbf{B}_{00} \cdot \tilde{\mathbf{B}}_{22} + \tilde{p}_{22}) + \tilde{B}_1^{y^2} - \tilde{B}_1^2, \quad (\text{A } 22)$$

$$\delta^2 W_{00}^{\text{Pl}}(\boldsymbol{\varphi}, \boldsymbol{\chi}) = (1/2) \int_{\text{Pl}_{00}} [\mathbf{B}_{10}(\boldsymbol{\varphi}) \cdot \mathbf{B}_{10}(\boldsymbol{\chi}) + \mathbf{j}_{00} \cdot (\boldsymbol{\varphi} \times \mathbf{B}_{10}(\boldsymbol{\chi})) + \boldsymbol{\chi} \cdot \nabla p_{00} \nabla \cdot \boldsymbol{\varphi} + (5/3) p_{00} \nabla \cdot \boldsymbol{\varphi} \nabla \cdot \boldsymbol{\chi}] d\tau, \quad (\text{A } 23)$$

$$\delta^2 W_0^{\text{Pl}}(\boldsymbol{\varphi}, \boldsymbol{\varphi}) = (1/2) \int_{\text{Pl}_0} [\mathbf{B}_1^2(\boldsymbol{\varphi}, \mathbf{B}_0) + \mathbf{j}_0 \cdot (\boldsymbol{\varphi} \times \mathbf{B}_1(\boldsymbol{\varphi}, \mathbf{B}_0)) - \mathbf{p}_1(\boldsymbol{\varphi}, p_0)] d\tau, \quad (\text{A } 24)$$

$$\delta^2 W_0^s(\boldsymbol{\varphi}, \boldsymbol{\varphi}) = (1/2) \int_{S_0} (\boldsymbol{\varphi} \cdot \mathbf{n}_0)^2 + \mathbf{n}_0 \cdot \nabla (B_0^2/2 - B_0^2/2 - p_0) dS. \quad (\text{A } 25)$$

## Appendix II: Example for the Connection Between Different Surface Representations

We consider the example of waterwaves and assume that the unperturbed watersurface is given by  $z \equiv 0$ . A scalar representation of the perturbed surface is

$$z = \zeta(x, y, t). \quad (\text{A } 26)$$

With  $x(t)$ ,  $y(t)$ ,  $z(t)$  being the coordinates of a fluid element on the surface, from (A 26) we get the well known relation

$$v_z = \dot{z} = \mathbf{v}_\perp \cdot \nabla \zeta + \partial \zeta / \partial t, \quad (\text{A } 27)$$

where

$$\mathbf{v}_\perp = \dot{x} \mathbf{e}_x + \dot{y} \mathbf{e}_y. \quad (\text{A } 28)$$

Using (A 26), the vectorial representation (14) reads

$$x \mathbf{e}_x + y \mathbf{e}_y + \zeta \mathbf{e}_z = \mathbf{r}_0 + \boldsymbol{\xi}(\mathbf{r}_0, t), \quad (\text{A } 29)$$

where  $\mathbf{r}_0$  is a point in the surface  $z \equiv 0$ . From (A 29) we get

$$\begin{aligned} \zeta_z(x_0, y_0, 0, t) \\ = \zeta(x_0 + \zeta_x(\mathbf{r}_0, t), y_0 + \zeta_y(\mathbf{r}_0, t), 0, t). \end{aligned} \quad (\text{A } 30)$$

Using (13) and (A 28), differentiation of (A 30) with respect to  $t$  yields again (A 27).

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